

QUANTUM VARIANCE ON QUATERNION ALGEBRAS, I

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ABSTRACT. We determine the quantum variance of a sequence of families of automorphic forms on a compact quotient arising from a non-split quaternion algebra. Our results compare to those obtained on $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R})$ in work of Luo, Sarnak and Zhao, whose method required a cusp. Our method uses the theta correspondence to reduce the problem to the estimation of metaplectic Rankin–Selberg convolutions. We apply it here to the first non-split case.

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1. INTRODUCTION

1.1. Overview. Let M be a negatively curved compact Riemannian manifold with $d := \dim(M) \geq 2$. It is known in various senses that a *random* Laplace eigenfunction on M of large eigenvalue is uniformly distributed, but very rarely known just *how* uniformly. To explain, denote by μ the probability measure on M that is a multiple of the volume measure and let $T \rightarrow \infty$ be a positive parameter. For each $t \in [T, 2T]$, choose an orthonormal basis for the t^2 -eigenspace of the Laplacian acting on $L^2(M, \mu)$. Let \mathcal{F}_T denote the union of these bases; it is finite set of cardinality $|\mathcal{F}_T| \asymp T^d$. To each $\varphi \in \mathcal{F}_T$, attach the L^2 -mass

$$\mu_\varphi := |\varphi|^2 \mu, \quad \mu_\varphi(\Psi) := \int_{x \in M} \Psi(x) |\varphi|^2(x) d\mu(x).$$

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It is a probability measure on M . A consequence of the quantum ergodicity theorem is that the *mean* $|\mathcal{F}_T|^{-1} \sum_{\varphi \in \mathcal{F}_T} \mu_\varphi$ converges weakly to μ . The *quantum variance problem* concerns the asymptotic determination of the quadratic form

$$\begin{aligned} \mathcal{V}_T(\Psi) &:= \frac{1}{|\mathcal{F}_T|} \sum_{\varphi \in \mathcal{F}_T} |\mu_\varphi(\Psi) - \mu(\Psi)|^2 \\ &= \frac{1}{|\mathcal{F}_T|} \sum_{\varphi \in \mathcal{F}_T} \int_{x, y \in M} \Psi(x) \overline{\Psi(y)} (|\varphi(x)|^2 - 1) (|\varphi(y)|^2 - 1) d\mu(x) d\mu(y) \end{aligned} \quad (1)$$

and its bilinearization $\mathcal{V}_T(\Psi_1, \Psi_2)$, which quantify the *correlations of the fluctuations* of the μ_φ . They extend naturally to the phase space $\mathbf{X} = S^*M$ by replacing μ_φ with its microlocal lift $\omega_\varphi(\Psi) := \langle \text{Op}(\Psi)\varphi, \varphi \rangle$ and μ with the Liouville measure.

Zelditch introduced these sums and showed using semiclassical techniques that $\mathcal{V}_T(\Psi) = O(1/\log T)$ (see [71, 72, 62]). A prediction of Feingold–Peres (see [18], [70, §15.6], [52, §4.1.3]) suggests that for “generic” M , the renormalized quantum variance $T^{d-1}\mathcal{V}_T$ is asymptotic to the classical variance of the geodesic flow. The quantum variance problem is thus extremely delicate: a solution in the expected form

$$T^{d-1}\mathcal{V}_T(\Psi) = \mathcal{V}_\infty(\Psi) + o(1) \quad (3)$$

demands not only the enormous improvement $\mathcal{V}_T(\Psi) = O(T^{1-d})$ upon the semiclassical estimate, but the further extraction of a main term.

Luo–Sarnak [43] insightfully observed that on the modular curve $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$, the quantum variance problem for Hecke–Laplace eigenfunctions may be profitably attacked using arithmetic techniques stemming from the theory of Hecke operators. The solution, obtained in sharpest form by Sarnak–Zhao [61] after the increasingly sophisticated works [41, 32, 42, 43, 73], is arithmetically interesting: central values of L -functions quantify the deviation of the generic prediction from the truth.

This article introduces a method for determining the quantum variance of families of automorphic forms on *compact* arithmetic quotients attached to *non-split* quaternion algebras B . The *non-compact* quotient $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ considered by Luo–Sarnak–Zhao arises algebraically from the *split* quaternion algebra $B = M_2(\mathbb{Q})$. The problem had been open in every non-split case prior to this work. We aim here to introduce our method by application to the simplest non-trivial non-split case.

The first step in the method of Luo–Sarnak–Zhao reduces the expression (2), which is *quadrilinear* in φ , to a complicated *bilinear* expression to which trace formulas apply and from which a colossal limiting bilinear form eventually emerges. This is achieved by combining the close relation between Fourier coefficients at the cusp ∞ of $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ and Hecke eigenvalues $\lambda(n)$ with the Hecke multiplicativity $\lambda(m)\lambda(n) = \sum_{d|m, n} \lambda(mn/d^2)$. Unfortunately, such an approach is unavailable on compact arithmetic quotients, which have no cusps and hence no notion of Fourier expansion that interacts suitably with the Hecke theory.

In our method, the theta correspondence is decisive. The well-known seesaw diagram

$$\begin{array}{ccc} \text{O}(B) & & \text{Mp}_2 \times \text{Mp}_2 \\ | & \searrow & | \\ \text{O}(1) \times \text{O}(B^0) & & \text{SL}_2 \end{array} \quad (4)$$

substitutes for the linearizing role played by Fourier coefficients and Hecke multiplicativity in the method of Luo–Sarnak–Zhao. The basic input powering convergence to the limit is decay of matrix coefficients for the dihedral¹ spectrum of SL_2 . The Rallis inner product formula and the Maass–Shintani–Waldspurger theta lift are shown to give natural interpretations to the main terms obtained here and in earlier works. We discover “secondary main terms” to the quantum variance sums at the square-root cancellation threshold. The primary novelty is that we prove asymptotic formulas for quantum variance sums on a non-split quaternion algebra for the first time.²

A general tool available on any quaternionic congruence quotient $\Gamma \backslash \mathbb{H}$ is the *triple product formula* (see [25, 67, 29]). When both ϕ and Ψ are Hecke eigenfunctions, it relates $|\mu_\varphi(\Psi)|^2$ to the L -value $L(\varphi \times \varphi \times \Psi, 1/2)$ and $\mathcal{V}_T(\Psi)$ to a weighted average whose analysis succumbs to well-developed techniques (see [43, 73, 61]) giving Lindelöf-consistent asymptotics of the form (3). Unfortunately, the triple product formula is non-linear; it does not apply when, for instance, Ψ is the sum of two *inequivalent* mean zero Hecke eigenfunctions Ψ_1, Ψ_2 . For the correlations

$$\begin{aligned} \mathcal{V}_T(\Psi_1, \Psi_2) &= \frac{1}{|\mathcal{F}_T|} \sum_{\varphi \in \mathcal{F}_T} \mu_\varphi(\Psi_1) \overline{\mu_\varphi(\Psi_2)} \\ &= \frac{1}{|\mathcal{F}_T|} \sum_{\varphi \in \mathcal{F}_T} \int_{x, y \in M} \Psi_1(x) \overline{\Psi_2(y)} |\varphi(x)|^2 |\varphi(y)|^2 \end{aligned}$$

one then expects (following [43]) the further cancellation $\mathcal{V}_T(\Psi_1, \Psi_2) = o(1/T)$ coming from the *independence of variation in sign* of the quantities $\mu_\varphi(\Psi_1), \mu_\varphi(\Psi_2)$ beyond the prediction $\mu_\varphi(\Psi_i) = O(T^{-1/2+o(1)})$ of the Lindelöf hypothesis for their *magnitude*. Detecting such cancellation is the fundamental difficulty overcome in the works of Luo–Sarnak–Zhao. It is achieved by lengthy analysis of symmetry properties of their emergent bilinear form, which is found (remarkably) to be Hecke self-adjoint. The analogous difficulty in our approach is ultimately addressed by the unramified case of the local theta correspondence for $(\mathrm{Mp}_2, \mathrm{O}(B^0))$.

Our main result concerns automorphic forms with ramification varying in a *non-archimedean* aspect. The setup differs superficially from what was discussed above, but retains the fundamental difficulty while allowing us to introduce the core ideas of the method as accessibly as we can.

1.2. Statement of main result. Terminology to follow is standard, and will be reviewed more thoroughly in §2. Set $G := \mathrm{GL}_2(\mathbb{Q}_2)$, $K := \mathrm{GL}_2(\mathbb{Z}_2)$. We fix a discrete cocompact subgroup $\Gamma < G$ arising from a maximal order in the quaternion algebra ramified at $\{\infty, 23\}$ (see §2.1). (The significance of the pair of numbers $(2, 23)$ is that it is the lexicographically smallest for which the problem to be discussed has all essential features; we have focused our discussion in this way to simplify notation and exposition.) Let N be a positive integral parameter tending off to ∞ . We consider the family \mathcal{F}_N of L^2 -normalized newforms (see §2.5) on a sequence of congruence covers

$$\mathbf{Y}_N := \Gamma \backslash G / K_{-N..N} \text{ of } \mathbf{Y} := \Gamma \backslash G / K,$$

¹In this article, only “split dihedral” forms (i.e., unitary Eisenstein series) contribute.

²We refer to [36, 35] for some analogous results in a simpler arithmetic setting.

where more generally

$$K_{-N_1 \dots N_2} := K \cap \begin{pmatrix} \mathbb{Z}_2 & 2^{N_1} \mathbb{Z}_2 \\ 2^{N_2} \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}. \quad (5)$$

The sets $\Gamma \backslash G, \mathbf{Y}_N, \mathbf{Y}$ come equipped with compatibly-defined Hecke correspondences T_n for odd integers $n \geq 1$ (see §2.2); on \mathbf{Y} , T_n is defined for all $n \geq 1$.

The sets \mathbf{Y} and $\Gamma \backslash G$ are roughly 2-adic analogues of a compact arithmetic hyperbolic surface and its cotangent bundle, respectively. The base space \mathbf{Y} turns out to have cardinality three. One may regard it as a $(2+1)$ -regular directed multigraph on three vertices and its cover \mathbf{Y}_N as (with suitable interpretation, taking into account torsion) the set of non-backtracking paths $y = (y_{-N} \rightarrow \dots \rightarrow y_N)$ of length $2N+1$ on \mathbf{Y} , with the covering map $\mathbf{Y}_N \rightarrow \mathbf{Y}$ given by the projection $y \mapsto y_0$. One may also identify \mathbf{Y} with the set of isomorphism classes of supersingular elliptic curve E in characteristic 23 (see [23, §2]) and \mathbf{Y}_N with the cover obtained by considering level structure consisting of pairs C_1, C_2 of cyclic subgroups of order 2^N with $C_1 \cap C_2 = \{0\}$; in fact, it will be convenient to adopt this algebraic perspective for some calculations (§9).

The space $\Gamma \backslash G$ has a natural invariant measure (see §2.4) which induces a natural measure on \mathbf{Y} . The space $L^2(\mathbf{Y})$ is three-dimensional, and decomposes as

$$L^2(\mathbf{Y}) = \mathbb{C} \oplus \mathbb{C}\Psi_1 \oplus \mathbb{C}\Psi_2,$$

where Ψ_1, Ψ_2 are orthonormal mean zero real-valued Hecke eigenfunctions, well-defined up to permutation and sign. For orientation, we record that under the Eichler/Jacquet–Langlands correspondence, Ψ_1, Ψ_2 correspond to the weight 2 newforms $\Psi_1^{\text{JL}}, \Psi_2^{\text{JL}}$ on $\Gamma_0(23)$ and \mathcal{F}_N to the set of weight 2 newforms on $\Gamma_0(2^{2N} \cdot 23)$.

To each $\varphi \in \mathcal{F}_N$ we attach the “harmonic weight” $\iota_\varphi := L^{(2)}(\text{ad } \varphi, 1)$ of size $2^{o(N)}$ (see §2.8); these mild weights play a role (non-obviously) similar to that in [61] and earlier works, and should be ignored on a first reading.

The fundamental object of study in this article is the real symmetric 2×2 matrix $V_N := (V_N^{kl})_{k,l=1,2}$ with entries

$$V_N^{kl} := \frac{1}{2^{2N}} \sum_{\varphi \in \mathcal{F}_N} \iota_\varphi \int_{x,y \in \Gamma \backslash G} \Psi_k(x) \Psi_l(y) |\varphi|^2(x) |\varphi|^2(y).$$

Equivalently, let μ_φ denote the measure on \mathbf{Y} given by $\mu_\varphi(\Psi) := \int_{\Gamma \backslash G} \Psi |\varphi|^2$; then

$$V_N^{kl} = \frac{1}{2^{2N}} \sum_{\varphi \in \mathcal{F}_N} \iota_\varphi \mu_\varphi(\Psi_k) \mu_\varphi(\Psi_l).$$

For the discussion of §1.1 specialized to the case of surfaces, the numerical analogue here of the quantity T is 2^N ; for instance, the family cardinality is $|\mathcal{F}_N| \asymp 2^{2N}$ (see §9.5 for an exact formula). To justify interpreting V_N^{kl} as a variance, we verify in §9.5 that

$$\frac{1}{|\mathcal{F}_N|} \sum_{\varphi \in \mathcal{F}_N} \mu_\varphi = \mu \quad (6)$$

with μ the probability measure on \mathbf{Y} that is a multiple of the pushforward of $\int_{\Gamma \backslash G}$. The entries of V_N quantify the correlations of the fluctuations of the L^2 -masses μ_φ of a random newform $\varphi \in \mathcal{F}_N$ on the congruence cover \mathbf{Y}_N when tested against the mean zero Hecke eigenfunctions Ψ_1, Ψ_2 on the base space \mathbf{Y} . The setup is thus the natural 2-adic analogue of that in §1.1.

We normalize the Hecke eigenvalues of Ψ_1, Ψ_2 by $T_n \Psi_i = \sqrt{n} \lambda_{\Psi_i}(n) \Psi_i$ and the standard L -functions $L(\Psi_k, s)$ by analytic continuation of the Dirichlet series $\sum_{n \geq 1} \lambda_{\Psi_k}(n)/n^s$ from the half-plane $\operatorname{Re}(s) > 1$, so that $s = 1/2$ is the central point.

Theorem 1 (Main result).

(i) **Existence of the limiting quantum variance.**

The 2×2 matrix limit $\lim_{N \rightarrow \infty} 2^N V_N$ exists.

(ii) **Diagonalization and determination of the limit.**

That limit is given by³

$$V_\infty := \begin{pmatrix} \tilde{L}(\Psi_1, \frac{1}{2}) & 0 \\ 0 & \tilde{L}(\Psi_2, \frac{1}{2}) \end{pmatrix}$$

with $\tilde{L}(\Psi_k, \frac{1}{2}) := P(\lambda_{\Psi_k}(2))L(\Psi_k, \frac{1}{2})$, $P(x) := \pi^2(15 - 4\sqrt{2}x)/69$.

(iii) **Effective rate of convergence to the limit.**

$|2^N V_N^{kl} - V_\infty^{kl}| \leq CN 2^{-N}$ for some absolute effective $C > 0$.

Each assertion in Theorem 1 is highly non-trivial – for instance, the existence of the limit already implies somewhat more than the strong improvement $V_N^{kl} \ll 2^{-N}$ over the trivial bound $V_N^{kl} \ll 1$ – but the most novel assertion is that for $k \neq l$,

$$\lim_{N \rightarrow \infty} 2^N V_N^{kl} = 0 \tag{7}$$

(in the strong quantitative form $2^N V_N^{kl} = O(N 2^{-N})$)

which is the first of its kind in any non-split/cocompact setting and lies genuinely beyond existing methods such as Fourier expansions at cusps, trace formulas, and triple product formulas, which fail even to reduce its proof to a technical problem. One can interpret (7) as reflecting a non-obvious Hecke symmetry enjoyed by the fluctuations of the measures μ_φ .

1.3. Application to moments of (square roots of) L -functions. The values $\mu_\varphi(\Psi_k)$ are basic examples of triple product periods, which have been extensively studied.⁴ The triple product formula relates the *squares* (but not the *signs*) of these periods to the central values of triple product L -functions. Unlike in the work of Luo–Sarnak–Zhao, we do not use the triple product formula in the proof of Theorem 1; nevertheless, it is instructive to indicate briefly how our results translate thereunder. The formula in question (see [29, 51, 27]) has the shape

$$\iota_\varphi |\mu_\varphi(\Psi_k)|^2 = 2^{-N} \frac{\tilde{L}(\varphi \times \varphi \times \Psi_k, \frac{1}{2})}{L(\operatorname{ad} \varphi, 1)}$$

where $\tilde{L}(\varphi \times \varphi \times \Psi_k, \frac{1}{2}) := c_0(\Psi_k, \varphi) L(\varphi \times \varphi \times \Psi_k, \frac{1}{2})$ for some explicit nonnegative quantity $c_0(\Psi_k, \varphi)$ that we expect⁵ is uniformly bounded as φ varies. (The reader will lose little in what follows by ignoring the factor $c_0(\Psi_k, \varphi)$, whose explication

³One has $\{L(\Psi_1, \frac{1}{2}), L(\Psi_2, \frac{1}{2})\} \approx \{0.552, 0.450\}$ according to [38, 23.2.1a.0], [38, 23.2.1a.1]

⁴See for instance the articles [24, 5, 17, 7], which focus on definite quaternion algebras as we do here, or [51], whose considerations apply to the analogous periods obtained from newforms on the tower of covers $\Gamma_0(2^{2N}) \backslash \mathbb{H} \rightarrow \operatorname{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ of the modular curve.

⁵By [49, Thm 34], this expectation holds when φ is principal series. It should follow in general from the methods of [27]. One could rewrite this article with the role of $p = 2$ replaced by that of some odd prime. We expect then that $c_0(\Psi_k, \varphi)$ would be independent of Ψ_k, φ . This expectation holds when φ is principal series by [49, Thm 34].

is beyond the scope of this paper.) The Lindelöf hypothesis thus predicts that the *magnitude* of the real number $\mu_\varphi(\Psi_k)$ is typically $\approx 2^{-N/2}$, but says nothing about its *sign*. Theorem 1 tells us that the signs of $\mu_\varphi(\Psi_k)$ and $\mu_\varphi(\Psi_l)$ are *highly unbiased* for $k \neq l$; its diagonal and off-diagonal $k \neq l$ cases translate to

$$\frac{1}{2^{2N}} \sum_{\varphi \in \mathcal{F}_N} \frac{\tilde{L}(\varphi \times \varphi \times \Psi_k, \frac{1}{2})}{L(\text{ad } \varphi, 1)} = \tilde{L}(\Psi_k, \frac{1}{2}) + O\left(\frac{\log |\mathcal{F}_N|}{\sqrt{|\mathcal{F}_N|}}\right), \quad (8)$$

$$\frac{1}{|\mathcal{F}_N|} \sum_{\varphi \in \mathcal{F}_N} \frac{\sqrt{\tilde{L}(\varphi \times \varphi \times \Psi_k, \frac{1}{2})} \sqrt{\tilde{L}(\varphi \times \varphi \times \Psi_l, \frac{1}{2})}}{L(\text{ad } \varphi, 1)} = O\left(\frac{\log |\mathcal{F}_N|}{\sqrt{|\mathcal{F}_N|}}\right) \quad (9)$$

with the choice of square-root in (9) given by the sign of the real number $\mu_\varphi(\Psi_k)$. The diagonal estimate (8) is consistent with the Lindelöf hypothesis on average.⁶ The off-diagonal estimate (9) represents *square-root cancellation beyond the Lindelöf hypothesis*, and does not follow from standard conjectures such as GRH. We discuss the error terms further in §6.5. For orientation, we record that $|\mathcal{F}_N| \asymp 2^{2N}$ and typically $C(\varphi \times \varphi \times \Psi_k) \asymp 2^{4N}$.⁷

1.4. Application to quantum unique ergodicity. We touch briefly on the relationship of our results to the study of worst-case behavior of the measures μ_φ . It is expected that $\mu_{\varphi_N} \rightarrow \mu$ for any sequence of $\varphi_N \in \mathcal{F}_N$ with $N \rightarrow \infty$, i.e., that $\mu_{\varphi_N}(\Psi_k) = o(1) := o_{N \rightarrow \infty}(1)$ for $k = 1, 2$. This expectation is much weaker than the prediction $|\mu_{\varphi_N}(\Psi_k)| \leq 2^{-(1+o(1))N/2}$ of the Lindelöf hypothesis, but remains an open variant of the arithmetic quantum unique ergodicity conjecture as in [46, 51, 27] which can be shown to follow from subconvexity. The special case in which φ_N generates a principal series representation of $\text{GL}_2(\mathbb{Q}_2)$ was recently confirmed unconditionally in [49]. Theorem 1 and Chebyshev's inequality imply that the Lindelöf prediction is essentially sharp, that it holds for a density $1 - o(1)$ subset of \mathcal{F}_N , and that all but a density $O(|\mathcal{F}_N|^{-1/2+o(1)})$ subset of $\varphi_N \in \mathcal{F}_N$ satisfy $\mu_{\varphi_N} \rightarrow \mu$ as $N \rightarrow \infty$ (compare with [41, 42, 43]).

1.5. Method. We now indicate in high-level terms why is it natural to study the quantum variance problem using theta functions and the diagram (4); a concrete implementation of this discussion may be found in §5.

The arithmetic quotients \mathbf{X} of interest to us are parametrized by quaternion \mathbb{Q} -algebras B ; one can take for \mathbf{X} the adelic quotient $B^\times(\mathbb{Q}) \backslash B^\times(\mathbb{A})$ or a further quotient thereof. Given a pair of cusp forms Ψ_1, Ψ_2 on \mathbf{X} with trivial central character, we would like to understand quantum variance sums of the shape

$$\sum_{\varphi \in \mathcal{F}} \langle |\varphi|^2, \Psi_1 \rangle \langle \Psi_2, |\varphi|^2 \rangle \quad (10)$$

for some “nice enough family of automorphic forms” \mathcal{F} .

⁶A direct proof of (9) using the approximate functional equation, Petersson formula and Voronoi summation is heuristically straightforward; technical complications arise because of “weight 2.” The proof of Theorem 1 does not use the triple product formula.

⁷We remark also that by replacing the φ, Ψ_k with some modifications belonging to the automorphic representation that they generate, it should be possible to derive explicit analogues of (8), (9) without the factors $c_0(\Psi_k, \varphi)$. For instance, it should be possible to achieve this by leaving Ψ_k as it is and replacing φ by the 2-adic microlocal lift introduced in [49] when φ is principal series and by an analogous construction involving non-split tori when φ is supercuspidal.

Our first observation is that sums of this general shape are related via the theta correspondence to four-fold integrals of theta functions: Suppose given a pair of elementary theta functions θ_1, θ_2 and ternary theta lifts h_1, h_2 of the cusp forms Ψ_1, Ψ_2 . More precisely, $\theta_1, \theta_2, h_1, h_2$ are functions on $\mathrm{SL}_2(\mathbb{Q}) \backslash \mathrm{Mp}_2(\mathbb{A})$ attached to Schwartz–Bruhat functions ϕ'_1, ϕ'_2 on \mathbb{A} and ϕ''_1, ϕ''_2 on $B^0(\mathbb{A})$, where B^0 is the trace zero subspace of B . The Parseval formula and seesaw duality (4) then give identities roughly of the shape

$$\langle \theta_1 h_1, \theta_2 h_2 \rangle = \left(\begin{array}{c} \text{generalized quantum variance sums} \\ \text{tested against } \Psi_1, \Psi_2 \end{array} \right). \quad (11)$$

We explain this shortly (§1.6) in a “toy example” and in the body of this article very concretely.

The identity (11) (together with its extension to non-pure tensors $\phi_i = \sum_{\alpha} \phi'_{i,\alpha} \otimes \phi''_{i,\alpha}$) suggests a natural strategy for attacking the quantum variance problem. It is not *a priori* clear that this strategy should succeed, and its implementation requires further novelties:

- (1) The precise sums appearing on the RHS of (11) depend heavily upon the local data $\phi'_1, \phi'_2, \phi''_1, \phi''_2$. One must thus confront the *inversion problem* for a family \mathcal{F} of automorphic forms which consists of exhibiting local data ϕ_i so that the RHS of (11) resembles the quantum variance sums (10) over \mathcal{F} .
- (2) There is then the unprecedented *analytic problem* of asymptotically evaluating the integral of four-fold products of theta functions on the LHS of (11) as the local data vary; we address this by proving asymptotic formulas of the shape

$$\langle \theta_1 h_1, \theta_2 h_2 \rangle \approx \langle \theta_1, \theta_2 \rangle \langle h_1, h_2 \rangle. \quad (12)$$

The two problems are intertwined in that one does not expect the nice asymptotic behavior (12) for completely general variation of the local data.

1.6. Quantum variance sums with Eisenstein observables via L -functions.

We include this subsection to convey the flavor of the first step (11) of our method to readers having some familiarity with integral representations of L -functions, but not necessarily with the theta correspondence and seesaw duality.

We aim to illustrate (11) for a “toy problem” (unrelated to the main result of this article) involving the Eisenstein series E_s on $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$, defined for a complex parameter s by meromorphic continuation of the sums

$$E_s(z) := \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2 - \{(0,0)\} \\ \gcd(c,d)=1}} \frac{y^s}{|cz + d|^{2s}} = y^s + \cdots$$

For $\mathrm{Re}(s)$ large enough and up to (important) normalizing factors, Rankin–Selberg theory and Shimura’s symmetric square integral representation give

$$\begin{aligned} L(\varphi \times \varphi, s) &:= \zeta(2s) \sum_{n=1}^{\infty} \frac{|\lambda_{\varphi}(n)|^2}{n^s} \approx \mu_{\varphi}(E_s), \\ L(\mathrm{sym}^2 \varphi, s) &:= \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_{\varphi}(n^2)}{n^s} \approx \langle \theta \tilde{E}_s, \varphi \rangle, \end{aligned}$$

for some Jacobi theta function θ and half-integral weight Eisenstein series \tilde{E}_s . Combining these identities with the factorization⁸ $L(\varphi \times \varphi, s) = \zeta(s)L(\text{sym}^2 \varphi, s)$ gives with $\tilde{E}'_s := \zeta(s)\tilde{E}_s$ that

$$\mu_\varphi(E_s) \approx \langle \theta \tilde{E}'_s, \varphi \rangle. \quad (13)$$

Note that $\varphi \mapsto \langle \theta \tilde{E}'_s, \varphi \rangle$ is *linear*, unlike $\varphi \mapsto \mu_\varphi(E_s)$, and that the non-linear identity (13), like the triple product formula, applies only when φ is an eigenfunction.⁹

Consider now the pair of unitary Eisenstein series $\Psi_1, \Psi_2 := E_{1/2+it_1}, E_{1/2+it_2}$ attached to $t_1, t_2 \in \mathbb{R}$. Let \mathcal{F} be an orthonormal basis of eigenfunctions for the discrete spectrum of $L^2(\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H})$; write \dots for the corresponding contribution of the continuous spectrum, and ignore (for the purposes of this formal discussion) that most of the sums/integrals written below are divergent. From (13) and Parseval, one obtains with $h_1, h_2 := \tilde{E}'_{1/2+it_1}, \tilde{E}'_{1/2+it_2}$ that

$$\sum_{\varphi \in \mathcal{F}} \mu_\varphi(\Psi_1) \overline{\mu_\varphi(\Psi_2)} + \dots \approx \sum_{\varphi \in \mathcal{F}} \langle \theta h_1, \varphi \rangle \langle \varphi, \theta h_2 \rangle + \dots = \langle \theta h_1, \theta h_2 \rangle. \quad (14)$$

Formally, quantum variance sums equal integrals of four-fold products of half-integral weight theta functions.

The theta correspondence and seesaw duality give non-formal analogues (11) of (14); the point of this article is thus to determine quantum variance asymptotics by deriving some such analogues and then proving asymptotic formulas (12) for the inner products that arise.

1.7. Further perspectives. We conclude this introduction by noting some reasons to have anticipated an approach to the quantum variance problem using the theta correspondence and half-integral weight forms:

- (1) The quantum variance theorems of Luo–Sarnak–Zhao give new proofs that $L(\pi, \frac{1}{2}) \geq 0$ (see [43, p773, (2)]); earlier proofs as in [34, 33] used half-integral weight forms.
- (2) The Rallis inner product formula implies that the composition of the Maass–Shintani–Waldspurger lift (for fixed local data) with the Petersson inner product has similar properties to those of the limiting bilinear form identified in the works of Luo–Sarnak–Zhao; the present work confirms that this similarity is no coincidence.
- (3) Any progress on the quantum variance problem presupposes some progress towards the multiplicity one theorem; the known constructive proof of that, due to Eichler and Shimizu, used the theta correspondence much as we do here.

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⁸This is the special case $|\lambda(n)|^2 = \sum_{d|n} \lambda(n^2/d^2)$ of the Hecke multiplicatively exploited by Luo–Sarnak–Zhao.

⁹The analysis of triple product averages mentioned above exploits the related identity $L(\varphi \times \Psi, 1/2) = L(\text{sym}^2 \varphi \times \Psi, 1/2)L(\Psi, 1/2)$.

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2. PRELIMINARIES

We record definitions, some postponed from §1.

2.1. Generalities. Recall that $G := \mathrm{GL}_2(\mathbb{Q}_2)$, $K := \mathrm{GL}_2(\mathbb{Z}_2)$. Let B be the quaternion algebra ramified at $\{\infty, 23\}$. Let $R \leq B$ be a maximal order. Choose an embedding of B into the matrix algebra $M_2(\mathbb{Q}_2)$ under which R embeds in $M_2(\mathbb{Z}_2)$. Take for Γ the image of $R[1/2]^\times$ under this embedding. Then $\Gamma < G$ is a discrete cocompact subgroup. The quotient $\mathbf{X} := \Gamma \backslash G$ is compact.

The reduced norm and trace on B and its extensions are denoted nr, tr . A superscripted 0 denotes “trace zero subgroup.” A subscripted 2 denotes “2-adic completion.”

2.2. Hecke operators. For an odd positive integer n , the Hecke operator T_n acts on functions $\varphi : \mathbf{X} \rightarrow \mathbb{C}$ by $T_n \varphi(x) := \sum_{\alpha \in M_n/\Gamma} \varphi(\alpha^{-1}x)$ where $M_n := R[1/2] \cap \mathrm{nr}^{-1}(n\mathbb{Z}[1/2]^\times)$. The group G acts on such functions by right translation, and commutes with the Hecke operators.

2.3. Automorphic forms. Denote by $\mathcal{A}(\mathbf{X})$ the space of *smooth* functions $\varphi : \mathbf{X} \rightarrow \mathbb{C}$, i.e., those that are right invariant by some open subgroup of G . For a subgroup $S \leq G$, denote by $\mathcal{A}(\mathbf{X})^S$ the subspace of right S -invariant functions, or equivalently, those that factor through the quotient $\mathbf{X}/S = \Gamma \backslash G/S$. We may and shall identify $\mathcal{A}(\mathbf{X})^S$ with a space of functions on \mathbf{X}/S . The Hecke operators act on it.

2.4. Measures. Equip G with the Haar measure assigning volume 2 to K , and \mathbf{X} with the quotient measure, denoted simply $\int_{\mathbf{X}}$; define $L^2(\mathbf{X})$ with respect to that measure.

Recall that $\mathbf{Y} := \Gamma \backslash G/K = \mathbf{X}/K$. By the convention of §2.3, functions on \mathbf{Y} are identified with right K -invariant functions on \mathbf{X} . For each $E \in \mathbf{Y}$, choose a representative $g_E \in G$. Set $R_E := R[1/2] \cap g_E R_2 g_E^{-1}$; it is a maximal order in B . Set $w_E := \#R_E^\times/\mathbb{Z}^\times = (1/2)\#R_E^\times$. For $\Psi : \mathbf{Y} \rightarrow \mathbb{C}$, one then has $\int_{\mathbf{X}} \Psi = \sum_{E \in \mathbf{Y}} \Psi(E)/w_E$.

2.5. Families of balanced newvectors. Let N be a positive integer. Recall the definition (5) of $K_{-N_1..N_2}$. By the multiplicity one theorem and local newvector theory, there is a unique (up to signs) maximal subset $\mathcal{F}_N \subset \mathcal{A}(\mathbf{X})^{K_{-N..N}}$ with the properties:

- Each $\varphi \in \mathcal{F}_N$ is an eigenfunction for the Hecke operator T_n for all odd natural numbers n . We accordingly write $T_n \varphi = \sqrt{n} \lambda_\varphi(n) \varphi$.
- Each $\varphi \in \mathcal{F}_N$ generates an irreducible representation of G under right translation.

- Each $\varphi \in \mathcal{F}_N$ is real-valued and orthonormal, and any two $\varphi, \varphi' \in \mathcal{F}_N$ are orthogonal to one another.
- Each $\varphi \in \mathcal{F}_N$ is orthogonal to any function on \mathbf{X} that is $K_{-N_1..N_2}$ -invariant from some ordered pair (N_1, N_2) with $N_1 \leq N$ and $N_2 \leq N$ and $(N_1, N_2) \neq (N, N)$.

The multiplicity one theorem implies moreover that each $\varphi \in \mathcal{F}_N$ is determined by its system of Hecke eigenvalues $\lambda_\varphi(n)$. The family \mathcal{F}_N is analogous to (and in Hecke-equivariant bijection with; see §2.7) the set of normalized newforms of weight 2 on $\Gamma_0(2^{2N} \cdot 23)$.

2.5.1. Remark. We consider here the families \mathcal{F}_N arising from the “balanced” subgroups $K_{-N..N}$. Our method applies to the other subgroups, such as the unbalanced ones $K_{0..N}$ more commonly denoted “ $K_0(2^N)$ ”, but the results obtained are nicer for those considered here.

2.6. Conventions on modular forms. We denote by $z := x + iy$ a typical element of the upper half-plane and write $q := e^{2\pi iz}$. “Modular” always means “modular with respect to some congruence subgroup Γ' of $\Gamma_0(4)$.” We analytically normalize holomorphic modular forms Φ of weight $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ by the factor $y^{k/2}$, so that q -expansions read $\Phi(z) = y^{k/2} \sum a_n q^n$ and dilations $\Phi(z) \mapsto \Phi(az)$ for $a \in \mathbb{Q}_+^\times$ are unitary for the Petersson inner product, which we normalize by $\langle \Phi_1, \Phi_2 \rangle := \int_{z \in \Gamma' \backslash \mathbb{H}} \Phi_1(z) \overline{\Phi_2(z)} d\nu(z)$ for Γ' small enough in terms of Φ_1, Φ_2 and $\nu = \nu_{\Gamma'}$ the probability measure that is a multiple of $y^{-2} dx dy$. Thus, for instance, $\langle 1, 1 \rangle = 1$, regardless of the congruence quotient $\Gamma' \backslash \mathbb{H}$ on which the constant function 1 is regarded as living. Write $\|\Psi\| := \langle \Psi, \Psi \rangle^{1/2}$.

2.7. Eichler/Jacquet–Langlands lifts. For $N \geq 1$ and $\varphi \in \mathcal{F}_N$, set

$$\Phi_\varphi(z) := y \sum_{\substack{n \geq 1: \\ \gcd(n, 2) = 1}} \sqrt{n} \lambda_\varphi(n) q^n$$

It is known¹⁰ that Φ_φ defines a weight 2 newform on $\Gamma_0(2^{2N} \cdot 23)$ with Hecke eigenvalues $\sqrt{n} \lambda_\varphi(n)$ for odd natural numbers n .

2.8. Harmonic weights. Recall from §1 that $\iota_\varphi := L^{(2)}(\text{ad } \varphi, 1)$. It is known [26] that $\iota_\varphi = 2^{o(N)}$ as $N \rightarrow \infty$. By the theory of Eisenstein series,¹¹ $\|\Phi_\varphi\|^2$ is the residue as $s \rightarrow 1^+$ for some sufficiently divisible $M \in \mathbb{Z}_{\geq 1}$ of the integral

$$\int_{\Gamma_1(M) \backslash \mathbb{H}} \left(\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_1(M)} \text{Im}(\gamma z)^s |\Phi_\varphi|^2(z) y^{-2} dx dy \right)$$

which unfolds to $\int_{y=0}^\infty y^s \sum_n n |\lambda_\varphi(n)|^2 y^2 e^{-4\pi n y} y^{-2} dy$ and then simplifies to

$$\int_{y=0}^\infty y^{s+1} e^{-4\pi y} \frac{dy}{y} \sum_{\substack{n \in \mathbb{Z}_{\geq 1}: \\ (n, 2) = 1}} \frac{|\lambda_\varphi(n)|^2}{n^s} = \frac{\Gamma(s+1)}{(4\pi)^{s+1}} \zeta_{23}(s+1) \frac{L^{(S)}(\text{ad } \varphi, s) \zeta^{(S)}(s)}{\zeta^{(S)}(2s)}$$

¹⁰This can be deduced from results of Eichler (see [14], [53, §2]) and Atkin–Lehner [1, Thm 3 (iii)].

¹¹see for instance [30, p138], [40, §5.1]

with $S := \{2, 23\}$ and $\zeta_p(s) := (1 - p^{-s})^{-1}$, $\zeta^{(S)}(s) := \zeta(s) / \prod_{p \in S} \zeta_p(s)$. Taking residues and using that $L_{23}(\text{ad } \varphi, s) = \zeta_{23}(s+1)$, we obtain $\|\Phi_\varphi\|^2 = \kappa_1 \iota_\varphi$ with

$$\kappa_1 := \frac{1}{(4\pi)^2 \zeta^{(S)}(2) \zeta_2(1) \zeta_{23}(1)}. \quad (15)$$

Define $\kappa_0 > 0$ by requiring that $\kappa_1 = \kappa_0^{-2} 2^{-2}$.

3. DEFINITIONS OF SOME THETA FUNCTIONS

The purpose of this section is to define some specific weight $3/2$ cuspidal theta functions h_k ($k = 1, 2$) belonging to the Shintani–Waldspurger lifts of the automorphic forms $\Psi_k : \mathbf{Y} \rightarrow \mathbb{C}$ defined in §1. The precise definition of the h_k is not immediately enlightening; it is the output of local computations to be discussed later.

Recall that for each element E of the three-element set $\mathbf{Y} = \Gamma \backslash G / K = \mathbf{X} / K$ we have defined a representative group element $g_E \in G$ and a maximal order R_E (see §2.4). Set $S_E := \mathbb{Z} + 2R_E$. Recall the reduced norm and trace $\text{tr}, \text{nr} : R_E, S_E \rightarrow \mathbb{Z}$ and the trace zero subgroups R_E^0, S_E^0 . The latter are rank three lattices which we regard as ternary quadratic forms with respect to nr . Because B splits at 2, there are isomorphisms $R_E \otimes_{\mathbb{Z}} \mathbb{Z}_2 \xrightarrow{\cong} M_2(\mathbb{Z}_2)$ taking (tr, nr) to (tr, \det) and $S_E^0 \otimes_{\mathbb{Z}} \mathbb{Z}_2$ to $\left\{ \begin{pmatrix} a & 2b \\ 2c & -a \end{pmatrix} : a, b, c \in \mathbb{Z}_2 \right\}$. There are thus isomorphisms $\ell_E : \mathbb{Z}^3 \rightarrow S_E^0$ so that the quadratic forms $Q_E : \mathbb{Z}^3 \rightarrow \mathbb{Z}$, $Q_E(a, b, c) := \text{nr}(\ell_E(a, b, c))$ satisfy $Q_E(a, b, c) \equiv \det \begin{pmatrix} a & 2b \\ 2c & -a \end{pmatrix} \pmod{4}$. For each $E \in \mathbf{Y}$, define quadratic characters $\chi_1^E, \chi_2^E, \chi_3^E : S_E^0 / 2S_E^0 \rightarrow \{\pm 1\}$ by writing a given element $\beta \in S_E^0$ in coordinates $\beta = \ell_E(a, b, c)$ and setting $\chi_1^E(\beta), \chi_2^E(\beta), \chi_3^E(\beta) := (-1)^{b+c}, (-1)^{a+c}, (-1)^{a+b}$; some coordinate-free definitions of the χ_i^E are recorded in §9.3. For $k = 1, 2$, we define the theta function h_k by the equivalent formulas

$$\begin{aligned} h_k(16z) &:= \kappa_0 y^{3/4} \sum_{E \in \mathbf{Y}} \frac{\Psi_k(E)}{w_E} \sum_{\beta \in S_E^0} \sum_{i=1,2,3} \chi_i^E(\beta) q^{\text{nr}(\beta)} \\ &= \kappa_0 y^{3/4} \sum_{E \in \mathbf{Y}} \frac{\Psi_k(E)}{w_E} \sum_{a,b,c \in \mathbb{Z}} \{(-1)^{b+c} + (-1)^{a+c} + (-1)^{a+b}\} q^{Q_E(a,b,c)} \\ &= \kappa_0 y^{3/4} \sum_{D \geq 0} \mu_D(\Psi_k) q^D \end{aligned} \quad (16)$$

where in the final expression, μ_D is the measure on \mathbf{Y} given by

$$\begin{aligned} \mu_D(E) &:= w_E^{-1} \sum_{\beta \in S_E^0 : \text{nr}(\beta) = D} \sum_{i=1,2,3} \chi_i^E(\beta) \\ &= w_E^{-1} \sum_{(a,b,c) \in \mathbb{Z}^3 : Q_E(a,b,c) = D} \{(-1)^{b+c} + (-1)^{a+c} + (-1)^{a+b}\}. \end{aligned}$$

We record for future reference an equivalent definition of the h_k . Let $E \in \mathbf{Y}$ correspond to the identity coset, so that $R_E = R$ and $S_E = S := \mathbb{Z} + 2R$. We thereby obtain characters $\chi_i := \chi_i^E : S^0 \rightarrow \{\pm 1\}$. They extend by continuity to S_2^0 and then extend by zero to elements χ_i of the Schwartz–Bruhat space $\mathcal{S}(B_2^0)$.

Define $\phi'' \in \mathcal{S}(B_2^0)$ by

$$\phi''(\beta) := 2^{-3}\kappa_0 \sum_{i=1,2,3} \chi_i(4\beta). \quad (17)$$

Define the theta kernel $\theta'' : \mathbb{H} \times \mathbf{X} \rightarrow \mathbb{C}$ by

$$\theta''(z, g) := y^{3/4} \sum_{\beta \in R[1/2]^0} \phi''(g^{-1}\beta g) q^{\text{nr}(\beta)}.$$

It is modular of weight $3/2$ in the first variable. Recalling our measure normalizations (§2.4), we find easily that

$$h_k(z) = \int_{g \in \mathbf{X}} \Psi_k(g) \theta''(z, g).$$

We refer to [23, §12] for a lucid discussion of the basic properties of some theta functions defined by analogy to the h_k but using the simpler measures $\mu_D^0(E) := w_E^{-1} \#\{\beta \in S_E^0 : \text{nr}(\beta) = D\}$ instead of the μ_D .

4. COMPARISON WITH ARITHMETIC VARIANCE

Luo–Rudnick–Sarnak [40] determined the limiting variance of the family of measures on $\text{SL}_2(\mathbb{Z}) \backslash \text{SL}_2(\mathbb{R})$ defined by closed geodesics ordered by discriminant. They termed this the *arithmetic variance* (see also [39, 37]). It turns out to be remarkably close to the limiting quantum variance of microlocal lifts (see [61, Remark 2]).

The analogue in our setting concerns representation numbers of ternary quadratic forms. Recall the definition of the constant κ_0 from §2.8, that of the 2×2 matrix V^∞ from Theorem 1, and that of the measures μ_D from §4.

Theorem 2 (Arithmetic variance). *For $k, l \in \{1, 2\}$,*

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{0 < D < x} \frac{\mu_D(\Psi_k)}{D^{1/4}} \frac{\mu_D(\Psi_l)}{D^{1/4}} = \frac{2}{\kappa_0^2 (4\pi)^{-3/2} \Gamma(3/2)} V_\infty^{kl}.$$

Despite the superficial similarity that both involve limits of quadratic sums of measures on \mathbf{Y} , Theorem 2 is much simpler than Theorem 1: it follows as in [40, 39, 37] from the Rankin–Selberg method applied to h_k followed by a Tauberian argument and the inner product calculation

$$\langle h_k, h_l \rangle = 2V_\infty^{kl} \quad (18)$$

to be discussed shortly. The factor $(4\pi)^{-3/2} \Gamma(3/2)$ arises as $\int_{y=0}^{\infty} |y^{3/4} e^{-2\pi y}|^2 \frac{dy}{y}$.

The “coincidence” that the same limiting matrix V_∞ appears in Theorems 1 and 2 may be understood as an instance of the correspondence principle (compare with [40, §1.4.6]).

5. REDUCTION OF THE PROOF OF THE MAIN RESULT

We now reduce the proof of Theorem 1 to that of some independent assertions whose proofs may be studied in any order. Retain the notation and conventions of §4, particularly the definition (16) of h_k . Define the following Jacobi theta function, which is modular of weight $1/2$:

$$\theta(z) := y^{1/4} \sum_{m \in \mathbb{Z} : \gcd(m, 2) = 1} q^{m^2}.$$

5.1. The reduction. To prove Theorem 1, it suffices to show more precisely for $k, l \in \{1, 2\}$ and large enough N that

$$2^N V_N^{kl} = \int_{\Gamma' \backslash \mathbb{H}} \theta(z) h_k(2^{2N} z) \overline{\theta(z) h_l(2^{2N} z)} d\nu(z) \quad (19)$$

$$= \int_{\Gamma' \backslash \mathbb{H}} |\theta|^2(z) h_k \overline{h_l}(2^{2N} z) d\nu(z) \quad (20)$$

$$= \int_{\Gamma' \backslash \mathbb{H}} |\theta|^2 d\nu \int_{\Gamma' \backslash \mathbb{H}} h_k \overline{h_l} d\nu + O(N 2^{-N}) \quad (21)$$

$$= V_\infty^{kl} + O(N 2^{-N}) \quad (22)$$

where Γ' denotes a sufficiently small congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. (Recall from §2.6 that $d\nu(z) = d\nu_{\Gamma'}(z)$ is always a probability measure.) The above steps encapsulate the heart of our method. The novelty lies primarily in the algebraic input (19), the analytic input (21), and the overall framework of the argument.

5.2. Inner products of theta lifts. We first discuss the easy steps of §5.1, or more precisely, those that follow readily from extensive machinery developed by others. The second step (20) is trivial. The fourth step (22) holds in the more precise form

$$\|\theta\|^2 \langle h_k, h_l \rangle = V_\infty^{kl}. \quad (23)$$

The identities (18) and (23) are equivalent because $\|\theta\|^2 = 1/2$ (see §10.11, (37)). The proof of (23) divides according to whether $k = l$ or not. If $k \neq l$, so that $V_\infty^{kl} := 0$, the strong multiplicity one theorem on \mathbf{Y} furnishes an odd prime p for which $\lambda_{\Psi_k}(p) \neq \lambda_{\Psi_l}(p)$.¹² The local data defining the lift $\Psi_k \mapsto h_k$ is unramified at p , which is known by Eichler's commutation relations¹³ to imply that h_k is an eigenfunction of Shimura's T_{p^2} Hecke operator with eigenvalue proportional to $\lambda_{\Psi_k}(p)$. Since T_{p^2} is self-adjoint for the Petersson inner product, the vanishing $\langle h_k, h_l \rangle = 0$ follows. This argument parallels the verification in [43, 73, 61] of symmetry properties of the limiting bilinear form arising from the off-diagonal Kuznetsov terms. Both arguments require a sort of “fundamental lemma,” given here by the unramified case of the local theta correspondence.

The $k = l$ case of (23) specializes an inner product formula whose paradigm was introduced by Rallis [59]. It is now known in great generality thanks to the work of several authors (see [19, 20] and references). See §12 for further discussion.

5.3. The algebraic input. We turn to the more difficult steps of §5.1. The identity (19), to be proved in §7, may be understood roughly as the synthesis of

- (1) the Parseval formula on $\Gamma' \backslash \mathbb{H}$;
- (2) an explicit seesaw identity proved using a weighted pretrace formula;
- (3) the local computation of the pushforward of a theta kernel, proved here using geometric arguments involving the Bruhat–Tits tree.

¹²SAGE confirms that one may take $p = 3$.

¹³see for instance [23, Prop 12.10]

5.4. The analytic input. The estimate (21) should be compared with a standard consequence of the spectral theorem: for *square-integrable* automorphic functions f_1, f_2 ,

$$\int_{\Gamma' \backslash \mathbb{H}} f_1(z) f_2(2^{2N} z) d\nu(z) = \int_{\Gamma' \backslash \mathbb{H}} f_1 d\nu \int_{\Gamma' \backslash \mathbb{H}} f_2 d\nu + O(N 2^{-(1-2\vartheta)N} \|f_1\| \|f_2\|) \quad (24)$$

with $\vartheta \in [0, 7/64]$ the best known bound for the Hecke eigenvalues of Maass cusp forms (and unitary Eisenstein series) at the prime 2. The hypotheses of (24) do not apply to the setup of (21) because $|\theta|^2 \notin L^2$. This problem is addressed by a regularized spectral decomposition of $|\theta|^2$ which gives the stronger estimate (21) – like (24) but with “ $\vartheta := 0$ ” – by showing that $|\theta|^2$ is orthogonal to every cusp form. See §10 and [50].

6. FURTHER REMARKS

6.1. The proof by [40, 61] that the arithmetic and quantum variance on the modular curve $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R})$ are related reduces to an explicit evaluation of each obtained by very different means; see [61, Remark 2]. By contrast, the first three steps of §5.1 provide a direct mechanism linking their analogues here.

6.2. Weights like the ι_φ (see §2.8) arose in the works of Luo–Sarnak–Zhao from an application of the Petersson/Kuznetsov formulas. We do not use such formulas here. The weights arise in the proofs of those formulas for the same reason they arise in our treatment; see §7. The weights ι_φ could be removed in our treatment by the technique of [61], but the statements and proofs of our results are simplified by retaining them.

6.3. The spectral identity resulting from the rearrangement (20) may be understood, in the language of Reznikov [60], as arising from the strong Gelfand configuration formed by the various diagonal embeddings of $\mathrm{Mp}_2 \times \mathrm{Mp}_2$ inside $\mathrm{Mp}_2 \times \mathrm{Mp}_2 \times \mathrm{Mp}_2 \times \mathrm{Mp}_2$, but for one caveat: local multiplicity one fails for some of the trilinear functionals on Mp_2 that we consider (implicitly). Compare with [45, §1.1.3].

6.4. The error bound in part (iii) of Theorem 1 may be written $O(|\mathcal{F}_N|^{-1/2} \log |\mathcal{F}_N|)$. The analogous bounds obtained by Luo–Sarnak–Zhao (see [43, Thm 1], [73, Thm 2], [61, §4]) are $O(|\mathcal{F}|^{-1/4+\varepsilon})$, roughly half as strong in the exponent as that obtained here. The present improvement seems to be a feature of the method rather than of the specific aspect considered. The quantitative strength of our method is of secondary importance; what matters most is that we have proven qualitative assertions as in (7) in a non-split setting for the first time.

6.5. The error bound in part (iii) of Theorem 1 is not optimal: by taking into account that the Ramanujan-type bound $2^{Nit} + 2^{(N-2)it} + \dots + 2^{-Nit} = O(N)$ for the 2^{2N} th normalized Hecke eigenvalue of the unitary Eisenstein series is rarely sharp, one can refine $N 2^{-N}$ down to 2^{-N} . In particular, (8) and (9) should hold with errors $O(|\mathcal{F}_N|^{-1/2})$; see §10.19 for some details. In principle, our method permits determination of “secondary main terms” $V_{kl}^{\infty 2}$ for which

$$2^N V_{kl}^N = V_{kl}^\infty + 2^{-N} V_{kl}^{\infty 2} + O(N^{-1} 2^{-N}),$$

or perhaps even asymptotic expansions up to $O(N^{-A}2^{-N})$ and beyond; such refinements are well beyond the scope of this paper. The quantities $V_{kl}^{\infty 2}$ should admit numerical evaluation. We expect (but have not checked) that $V_{kl}^{\infty 2} \neq 0$ for all k, l ; if so, then the estimate $2^N V_{kl}^N = V_{kl}^\infty + O(2^{-N})$ is best possible in that it cannot be improved to $2^N V_{kl}^N = V_{kl}^\infty + o(2^{-N})$.

6.6. It would be interesting to

- (1) understand any sense in which Theorem 1 holds after passing to algebraic parts modulo a suitable prime λ of $\overline{\mathbb{Q}}$ (cf. [55, 56, 16]);
- (2) consider $(p, 23)$ for an odd prime p instead of $(2, 23)$;
- (3) shrink the family \mathcal{F}_N as in [42], perhaps using [49, Thms 17 and 33];
- (4) implement the refinements suggested in §6.5, §10.19.

7. PROOF OF THE ALGEBRAIC INPUT

We now prove (19). Our argument is inspired by those of Eichler [15] and Shimizu [64] as well as Gross's formulation [23, (12.13)] of Eichler's trace formula.

7.1. Weighted pretrace formula. A convolution kernel $f \in C_c^\infty(G)$ acts on an automorphic form $\varphi \in \mathcal{A}(\mathbf{X})$ by the regular representation $\rho_{\text{reg}}(f)\varphi(x) := \int_{g \in G} \varphi(xg)f(g) = \int_{y \in \mathbf{X}} \varphi(y)k_f(x, y)$ with kernel k_f having the geometric and spectral expansions

$$\sum_{\gamma \in \Gamma} f(x^{-1}\gamma y) = k_f(x, y) = \sum_{\varphi \in \mathcal{B}(\mathcal{A}(\mathbf{X}))} \overline{\varphi(x)}(\rho_{\text{reg}}(f)\varphi)(y).$$

For nonnegative integers N_1, N_2 , denote by $e_{-N_1..N_2}$ the multiple of the characteristic function of $K_{-N_1..N_2}$ for which $\int_G e_{-N_1..N_2} = 1$. Thus $\rho_{\text{reg}}(e_{-N_1..N_2})$ defines the orthogonal projection onto $\mathcal{A}(\mathbf{X})^{K_{-N_1..N_2}}$. Set

$$f := e_{-N..N} - e_{-N+1..N} - e_{-N..N-1} + e_{-N+1..N-1} \in C_c^\infty(G).$$

It was shown in [48] that for $N \geq 2$, $\rho_{\text{reg}}(f)$ defines the orthogonal projection onto the span of the orthonormal set \mathcal{F}_N defined in §2.5. Thus

$$\sum_{\varphi \in \mathcal{F}_N} \overline{\varphi(x)}\varphi(y) = \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y).$$

For an odd natural number n , we apply the Hecke operator T_n to both sides and restrict to $x = y =: g$, giving

$$\sum_{\varphi \in \mathcal{F}_N} \sqrt{n}\lambda_\varphi(n)|\varphi|^2(g) = \sum_{\gamma \in M_n} f(g^{-1}\gamma g).$$

We fix $k \in \{1, 2\}$ and integrate against Ψ_k to obtain¹⁴

$$\sum_{\varphi \in \mathcal{F}_N} \sqrt{n}\lambda_\varphi(n)\mu_\varphi(\Psi_k) = \int_{g \in \mathbf{X}} \Psi_k(g) \sum_{\gamma \in M_n} f(g^{-1}\gamma g). \quad (25)$$

¹⁴The works [69, 31, 2, 3] also consider such weighted trace formulas.

7.2. Introduction of theta functions. The discussion thus far has been general: it applies with inessential modification to any compact quotient $\Gamma \backslash G$ involving any group G . Recalling now that $\Gamma \backslash G$ arose from the multiplicative structure on a quaternion algebra B , we prepare to exploit the *additive* structure of B . Set $B_2 := B \otimes_{\mathbb{Q}} \mathbb{Q}_2 \cong M_2(\mathbb{Q}_2)$. Denote by $\mathcal{S}(B_2)$ the Schwartz–Bruhat space, consisting of locally constant compactly supported functions. Define $\phi \in \mathcal{S}(B_2)$ as follows: For nonnegative integers N_1, N_2 , denote by $e'_{-N_1..N_2}$ the multiple of the characteristic function of the \mathbb{Z}_2 -order

$$\begin{pmatrix} \mathbb{Z}_2 & 2^{N_1}\mathbb{Z}_2 \\ 2^{N_2}\mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$$

for which $e'_{-N_1..N_2}|_{\text{nr}^{-1}(\mathbb{Z}_2^\times)} = e_{-N_1..N_2}$. Thus, e.g., $e'_{0..0}(1) = 1/2$, $e'_{-N..N}(1) = 3 \cdot 2^{2N-2}$. Set

$$\phi := e'_{-N..N} - e'_{-N+1..N} - e'_{-N..N-1} + e'_{-N+1..N-1} \in \mathcal{S}(B_2).$$

Thus ϕ is to orders as f is to their unit groups.¹⁵ Observe that $\text{nr}(\text{supp}(f)) \subseteq \mathbb{Z}_2^\times$ and that $n\mathbb{Z}[1/2]^\times \cap \mathbb{Z}_2^\times = \{n\}$. Thus if $\gamma \in M_n$ satisfies $f(g^{-1}\gamma g) \neq 0$, then $\text{nr}(\gamma) = n$. Moreover, $f(g^{-1}\gamma g) = \phi(g^{-1}\gamma g)$. Therefore

$$\sum_{\gamma \in M_n} f(g^{-1}\gamma g) = \sum_{\alpha \in R[1/2] \cap \text{nr}^{-1}(n)} \phi(g^{-1}\alpha g). \quad (26)$$

Summing (25), (26) over odd natural numbers n gives for $z \in \mathbb{H}$, $q := e^{2\pi iz}$ with¹⁶

$$\Theta(\phi, z, g, g) := y \sum_{\alpha \in R[1/2]} \phi(g^{-1}\alpha g) 1_{\mathbb{Z}_2^\times}(\text{nr}(\alpha)) q^{\text{nr}(\alpha)}$$

and Φ_φ as in §2.7 that

$$\sum_{\varphi \in \mathcal{F}_N} \mu_\varphi(\Psi_k) \Phi_\varphi(z) = \int_{g \in \mathbf{X}} \Psi_k(g) \Theta(\phi, z, g, g). \quad (27)$$

7.3. Pushforward of a theta kernel. The function Ψ_k is right K -invariant, so the RHS of (27) is unchanged by replacing ϕ with its average $\phi^K \in \mathcal{S}(B_2)$ defined by

$$\phi^K(b) := \frac{1}{\text{vol}(K)} \int_{s \in K} \phi(s^{-1}bs).$$

Proposition 3. *Let ϕ'' be as in §3. For $m \in \mathbb{Q}_2$ and $\beta \in B_2^0$, one has $\phi^K(m + \beta) = \kappa_0^{-1} 2^{2N-1} 1_{\mathbb{Z}_2}(m) \phi''(2^{-N}\beta)$.*

The proof of Proposition 3 is a local computation of what might be called “partial orbital integrals.” We postpone it to §8. We see now using the orthogonal decomposition $R[1/2] = \mathbb{Z}[1/2] \oplus R[1/2]^0$ that (cf. §3, §5)

$$\Theta(\phi^K, z, g, g) = c_N \theta(z) \theta''(2^{2N}z, g),$$

with $c_N := \kappa_0^{-1} 2^{2N-1} (2^{2N})^{-3/4}$, hence upon integrating against Ψ_k that

$$\sum_{\varphi \in \mathcal{F}_N} \mu_\varphi(\Psi_k) \Phi_\varphi(z) = c_N \theta(z) h_k(2^{2N}z). \quad (28)$$

¹⁵The choice $\phi := f$ would also work; the indicated choice of ϕ turns out to be computationally convenient. We remark that in an adelic setting, the analogous passage at almost all unramified places from unit groups to orders is responsible for the harmonic weights ι_φ .

¹⁶ Θ defines a variant of the diagonal restriction of the Eichler/Shimizu theta kernel.

7.4. Parseval. The multiplicity one theorem on \mathbf{X} and the self-adjointness of the Petersson inner product for the classical Hecke operators implies that $\Phi_\varphi, \Phi_{\varphi'}$ are orthogonal for $\varphi \neq \varphi'$. Recall from §2.8 that $\|\Phi_\varphi\|^2 = \kappa_1 \iota_\varphi$. For $k, l \in \{1, 2\}$, we obtain

$$\kappa_1 \sum_{\varphi \in \mathcal{F}_N} \iota_\varphi \mu_\varphi(\Psi_k) \mu_\varphi(\Psi_l) = |c_N|^2 \int_{z \in \Gamma' \backslash \mathbb{H}} \theta(z) h_k(2^{2N} z) \overline{\theta(z) h_l(2^{2N} z)} d\nu(z).$$

Dividing through by $2^N \kappa_1$ and verifying that $|c_N|^2 = 2^N \kappa_1$, we obtain (19).

7.5. Remark. Several authors¹⁷ have established explicit seesaw identities using strong multiplicity one and newvector theory on \mathbf{X} to write some unknown constant multiple of $\varphi \otimes \overline{\varphi}$ as an explicit theta lift $\int_{z \in \Gamma' \backslash \mathbb{H}} \overline{\Phi_\varphi(z)} \Theta(\phi, z, \cdot, \cdot) d\nu(z)$ and then seesaw duality to determine the constant. The approach developed above differs in that it avoids direct analysis of the theta lift from SL_2 to $\mathrm{O}(B)$; this was achieved by unfolding a small part of the proof of that case of the global theta correspondence. The present approach is far more direct for our purposes because of subtleties arising from oldforms; moreover, it generalizes to more complicated families and to archimedean aspects.

8. COMPUTATION OF PARTIAL ORBITAL INTEGRALS

In this section we carry out the local calculation (Proposition 3) postponed in §7.3. The shape of this calculation (specifically the “separation” of the factors $1_{\mathbb{Z}_2}(m)$ and $\theta''(2^{-N}\beta)$ by the multiplicative dilation 2^N) is crucial to the success of the method. On the other hand, it may be instructive to note that Proposition 3, and hence the contents of this section, become unnecessary if one is willing to settle for a weaker and (much) less natural variant of Theorem 1 involving sums of the shape

$$\sum_{\pi \subseteq \mathcal{A}(\mathbf{X})} \iota_\pi \sum_{\varphi_1, \varphi_2 \in \mathcal{B}(\pi^{K[N]})} \mu_{\varphi_1}(\Psi_k) \mu_{\varphi_2}(\Psi_l), \quad (29)$$

where π traverses the irreducible submodules, $\iota_\pi := L^{(2)}(\mathrm{ad} \pi, 1)$ and $\mathcal{B}(\pi^{K[N]})$ is an orthonormal basis for the vectors invariant by the principal congruence subgroup $K[N]$ of K consisting of those elements congruent modulo N to a scalar. One may understand the purpose of this section as to reduce the average within π in (29) to an individual newvector $\varphi \in \mathcal{F}_N \cap \pi$.

In fact, we will give two complementary proofs of Proposition 3:

- (1) In this section, we record a direct proof based on the matrix Fourier transform and analysis of conjugacy classes in $\mathrm{GL}_2(\mathbb{Z}_2)$; this first proof is computationally involved, but concrete.
- (2) In §9, we record a geometric proof involving the Bruhat–Tits tree; that second proof is free of computational difficulties, but requires additional setup.

¹⁷See for instance [8, §10], [4, p230], [67, Lemma 2], [6, §5], [54, §5.4], [55, §3.2]. See also [28, §11] and [3] for further variants of (27), (28) proved on *split* quotients by different means.

8.1. Reduction to an identity of functions on a finite matrix ring. Observe first that the identity in Proposition 3 holds in the special case $m = 1, \beta = 0$; indeed, both sides specialize to the same nonzero quantity $3 \cdot 2^{2N-4}$. It will thus suffice to verify that identity up to an unspecified constant multiple; this purely technical reduction frees us from worrying about proportionality factors in what follows.

Observe next that the identity in question is one of functions of the variable $b = m + \beta \in B_2$ supported on $b \in M_2(\mathbb{Z}_2)$ and invariant under translation by $M_2(2^N \mathbb{Z}_2)$; it is thus equivalent to an identity between functions on $M_2(\mathfrak{o})$, where $\mathfrak{o} := \mathbb{Z}_2/2^N \mathbb{Z}_2 \cong \mathbb{Z}/2^N \mathbb{Z}$. Expanding the definitions, we reduce to establishing the following explicit identity of functions on $M_2(\mathfrak{o})$:

Lemma 4. *Set $\varpi := 2 \hookrightarrow \mathfrak{o}$ and $\mathfrak{p} := \varpi \mathfrak{o}$. For integers $m < 0 < m'$, let $\eta_{m..m'} : M_2(\mathfrak{o}) \rightarrow \mathbb{C}$ denote the characteristic function of the subset $\begin{pmatrix} \mathfrak{o}^\times & \mathfrak{p}^{-m} \\ \mathfrak{p}^{m'} & \mathfrak{o}^\times \end{pmatrix}$ of $M_2(\mathfrak{o})$. Define $\Phi^0 : M_2(\mathfrak{o}) \rightarrow \mathbb{C}$ by*

$$\Phi^0 := \eta_{-N..N} - (1/2)\eta_{-N..N-1} - (1/2)\eta_{-N+1..N} + (1/4)\eta_{-N+1..N-1}.$$

Define $\Phi : M_2(\mathfrak{o}) \rightarrow \mathbb{C}$ by

$$\Phi(x) := \sum_{g \in \mathrm{GL}_2(\mathfrak{o})} \Phi^0(g^{-1}xg).$$

Let $\Phi' : M_2(\mathfrak{o}) \rightarrow \mathbb{C}$ denote the function supported on elements of the form

$$t = \begin{pmatrix} v + \varpi^{N-2}x & \varpi^{N-1}y \\ \varpi^{N-1}z & v - \varpi^{N-2}x \end{pmatrix} \text{ with } v \in \mathfrak{o}^\times \text{ and } x, y, z \in \mathfrak{o}$$

and given on such elements by

$$\Phi'(t) := (-1)^{x+y} + (-1)^{y+z} + (-1)^{x+z}.$$

Then the functions Φ, Φ' are constant multiples of one another.

8.2. Application of the Fourier transform. To prove (4), we use the Fourier transform on $M_2(\mathfrak{o})$. Let ζ be a primitive 2^N th root of unity. For $a \in \mathfrak{o}$, the quantity ζ^a is well-defined. For $f : M_2(\mathfrak{o}) \rightarrow \mathbb{C}$, define its Fourier transform $\mathcal{F}f : M_2(\mathfrak{o}) \rightarrow \mathbb{C}$ by the formula $\mathcal{F}f(x) := \sum_{y \in M_2(\mathfrak{o})} f(y) \zeta^{(x,y)}$, where $(x, y) := \det(x+y) - \det(x) - \det(y) = x_{11}y_{22} + x_{22}y_{11} - x_{12}y_{21} - x_{21}y_{12}$. The Fourier transform is equivariant for conjugation by $\mathrm{GL}_2(\mathfrak{o})$. It is also injective, since it satisfies an inversion formula. We compute the Fourier transform of Φ^0 by applying the inclusion-exclusion identity (for $-N \leq m \leq -1, 1 \leq m' \leq N$)

$$\eta_{m..m'} = 1 \begin{pmatrix} \mathfrak{o} & \mathfrak{p}^{-m} \\ \mathfrak{p}^{m'} & \mathfrak{o} \end{pmatrix} - 1 \begin{pmatrix} \mathfrak{p} & \mathfrak{p}^{-m} \\ \mathfrak{p}^{m'} & \mathfrak{o} \end{pmatrix} - 1 \begin{pmatrix} \mathfrak{o} & \mathfrak{p}^{-m} \\ \mathfrak{p}^{m'} & \mathfrak{p} \end{pmatrix} + 1 \begin{pmatrix} \mathfrak{p} & \mathfrak{p}^{-m} \\ \mathfrak{p}^{m'} & \mathfrak{p} \end{pmatrix}$$

followed by the Fourier identity

$$\mathcal{F}1 \begin{pmatrix} \mathfrak{p}^a & \mathfrak{p}^b \\ \mathfrak{p}^c & \mathfrak{p}^d \end{pmatrix} = 2^{a+b+c+d} 1 \begin{pmatrix} \mathfrak{p}^{N-d} & \mathfrak{p}^{N-c} \\ \mathfrak{p}^{N-b} & \mathfrak{p}^{N-a} \end{pmatrix} \text{ for } 0 \leq a, b, c, d \leq N$$

to see that

$$2^{-2N} \mathcal{F}\Phi^0 = \sigma_{N,N} - (1/2)\sigma_{N-1,N} - (1/2)\sigma_{N,N-1} + (1/4)\sigma_{N-1,N-1}, \quad (30)$$

where $\sigma_{n,n'}$ denotes the characteristic function of $S(n, n') := \begin{pmatrix} \mathfrak{p}^n & \mathfrak{o}^\times \\ \mathfrak{o}^\times & \mathfrak{p}^{n'} \end{pmatrix}$. By a similar calculation, we see that

$$\mathcal{F}\Phi' = ? \left(1_{\text{tr}^{-1}(\mathfrak{p}^N)} - \frac{1}{2} 1_{\text{tr}^{-1}(\mathfrak{p}^{N-1})} \right) 1_E$$

for some unimportant scalar $?$, where $1_{\text{tr}^{-1}(\mathfrak{p}^n)}$ denotes the characteristic function of $\{x \in M_2(\mathfrak{o}) : \text{trace}(x) \in \mathfrak{p}^n\}$ and $E := E_1 \sqcup E_2 \sqcup E_3$ with $E_i := X_i + \mathfrak{p}M_2(\mathfrak{o})$ and $X_1 := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $X_2 := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $X_3 := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Set $H := \{\text{GL}_2(\mathfrak{o}) : g \equiv 1(\mathfrak{p})\}$. Since the E_i are conjugate under $\text{GL}_2(\mathfrak{o})$ and $E_1 = S(1, 1)$, we reduce to verifying that

$$\sum_{g \in H} \mathcal{F}\Phi^0(g^{-1}xg) = ? \left(1_{\text{tr}^{-1}(\mathfrak{p}^N)} - \frac{1}{2} 1_{\text{tr}^{-1}(\mathfrak{p}^{N-1})} \right) 1_{S(1,1)}(x) \quad (31)$$

for all $x \in M_2(\mathfrak{o})$ and some scalar $?$ not depending upon x .

8.3. Summing over orbits. Consider the map $\kappa : S(1, 1) \rightarrow \mathfrak{p} \times \mathfrak{o}^\times$ given by $\kappa(X) := (\text{tr}(X), \text{nr}(X))$. By Hensel's lemma and an (omitted) orbit-stabilizer argument, the map κ is surjective, its fibers all have the same cardinality, and each fiber is an orbit for the conjugation action of H . It follows for $-0 < n, n' \leq N$ that $\kappa|_{S(n, n')}$ surjects onto $\mathfrak{p}^{\min(n, n')} \times \mathfrak{o}^\times$ with fibers of equal cardinality and H acting transitively on each fiber. With the notation $n_0 := \min(n, n')$ and $n_1 := \max(n, n')$, we deduce that $\sum_{g \in H} 1_{gS(n, n')g^{-1}}$ is a constant multiple of the characteristic function of $S(1, 1) \cap \text{tr}^{-1}(\mathfrak{p}^{n_0})$. Since $S(n, n')$ has $2^{4N-2-n-n'}$ elements and $S(1, 1) \cap \text{tr}^{-1}(\mathfrak{p}^{n_0})$ has 2^{4N-3-n_0} elements, we obtain

$$\sum_{g \in H} 1_{gS(n, n')g^{-1}} = |H| 2^{1-n_1} 1_{S(1,1) \cap \text{tr}^{-1}(\mathfrak{p}^{n_0})}.$$

Substituting this into (30), we deduce that the LHS of (31) is a constant multiple of the value taken at x by the function

$$\left(1_{\text{tr}^{-1}(\mathfrak{p}^N)} - \frac{1}{2} 1_{\text{tr}^{-1}(\mathfrak{p}^{N-1})} - \frac{1}{2} 1_{\text{tr}^{-1}(\mathfrak{p}^{N-1})} + \frac{2}{4} 1_{\text{tr}^{-1}(\mathfrak{p}^{N-1})} \right) 1_{S(1,1)}$$

which simplifies to the RHS of (31).

9. FLUCTUATIONS OF FIXED LINES

The main purpose of this section is to record the geometric proof of Proposition 3 promised in §8. We also verify the mean statistics (6) used to justify interpreting V_N as a variance.

9.1. Translation to a geometric problem. Recall the notation of §2. Set $S := \mathbb{Z} + 2R$. Retain the notation ϕ from §7.2.

The function ϕ and its conjugates under K are supported on $R_2 \cong M_2(\mathbb{Z}_2)$, and the order R is dense in R_2 , so our task reduces (for notational convenience) to determining $\phi^K(\alpha)$ for all α in R . Recall that we have fixed an embedding $R \hookrightarrow R_2 \cong M_2(\mathbb{Z}_2) = \text{End}(\mathbb{Z}_2^2)$. Denote by $E[2^\infty]$ the abelian group $(\mathbb{Q}_2/\mathbb{Z}_2)^2$. It is thus a module for R under left multiplication. It is a direct limit of the submodules $E[2^N] := (2^{-N}\mathbb{Z}_2/\mathbb{Z}_2)^2$.

Our notation reflects that R may be identified with the endomorphism ring $\text{End}(E)$ of a supersingular elliptic curve E in characteristic 23 (see [23, §2]) and the groups $E[2^N], E[2^\infty]$ defined above with the corresponding torsion subgroups.

We have found this perspective helpful in forming intuition for the calculations to follow.

For a pair of nonnegative integers $N_1, N_2 \geq 0$, denote by \mathcal{L}_{N_1, N_2} the set of ordered pairs (C_1, C_2) consisting of cyclic subgroups $C_1, C_2 \leq E[2^\infty]$ of respective orders $2^{N_1}, 2^{N_2}$ satisfying $C_1 \cap C_2 = \{0\}$. The group K acts transitively on \mathcal{L}_{N_1, N_2} . The subgroup $K_{-N_1 \dots N_2}$ is the stabilizer of some “standard” pair $(C_1, C_2) \in \mathcal{L}_{N_1, N_2}$ tailored to the standard basis. Recall from §2.4 that the Haar on G assigns volume 2 to K . It follows for $\alpha \in R$ that

$$\frac{1}{\text{vol}(K)} \int_{s \in K} e'_{-N_1 \dots N_2}(s^{-1} \alpha s) = (1/2) \text{Fix}_{N_1, N_2}(\alpha),$$

where $\text{Fix}_{N_1, N_2} : R \rightarrow \mathbb{Z}_{\geq 0}$ is given by

$$\text{Fix}_{N_1, N_2}(\alpha) := \#\{(C_1, C_2) \in \mathcal{L}_{N_1, N_2} : \alpha C_1 \leq C_1, \alpha C_2 \leq C_2\}. \quad (32)$$

Thus for $N \geq 1$,

$$\phi^K(\alpha) = (1/2) \text{Fix}_{N, N}^\#(\alpha) \quad (33)$$

where more generally for $N_1, N_2 \geq 1$,

$$\text{Fix}_{N_1, N_2}^\# := \text{Fix}_{N_1, N_2} - \text{Fix}_{N_1-1, N_2} - \text{Fix}_{N_1, N_2-1} + \text{Fix}_{N_1-1, N_2-1}.$$

Our task is thus equivalent to evaluating the functions $\text{Fix}_{N, N}^\#$:

Proposition 5. *Let $N \geq 2$. Let $\alpha \in R$ with $\text{Fix}_{N, N}^\#(\alpha) \neq 0$. Then $\alpha \in \mathbb{Z} \oplus 2^{N-2} S^0$. For $m \in \mathbb{Z}$ and $\beta \in S^0$,*

$$\text{Fix}_{N, N}^\#(m + 2^{N-2} \beta) = 2^{2N-3} \sum_{i=1,2,3} \chi_i(\beta). \quad (34)$$

The proof is given below. To see how Proposition 5 implies Proposition 3, write $\alpha = m + \beta$. By (33) and (34), we have $\phi^K(\alpha) = 2^{2N-4} 1_{\mathbb{Z}_2}(m) \sum_{i=1,2,3} \chi_i(2^{2-N} \beta)$. We conclude that $\phi^K(\alpha) = \kappa_0^{-1} 2^{2N-1} 1_{\mathbb{Z}_2}(m) \phi''(2^{-N} \beta)$ upon recalling the definition (17) of ϕ'' .

9.2. The tree. We record a realization of the Bruhat–Tits tree \mathcal{T} of $\text{PGL}_2(\mathbb{Q}_2)$ (see [66], [63], [13, §1.2]) relative to a basepoint and fix some terminology.

9.2.1. The *vertices* of \mathcal{T} are the cyclic subgroups L of $E[2^\infty] = (\mathbb{Q}_2/\mathbb{Z}_2)^2$. Two vertices L, L' are connected by an *edge* if one contains the other with index 2. The undirected graph \mathcal{T} is then a 3-regular tree.

9.2.2. The trivial subgroup $\{0\}$ belongs to \mathcal{T} ; we call it the *origin*. For $n \geq 0$, denote by $\mathcal{T}_n \subset \mathcal{T}$ the set of vertices at distance n from the origin; these are the cyclic 2^n -subgroups of $E[2^\infty]$. The nearest vertex to the origin in the convex hull of a pair of vertices L and L' is $L \cap L'$. We call two vertices L, L' *independent* if $L \cap L' = \{0\}$, or equivalently, if their convex hull contains the origin. The convex hull of L and the origin consists of those L' for which $L' \leq L$. The *children* of a vertex $v \in \mathcal{T}_n$ are those $v' \in \mathcal{T}_{n+1}$ neighboring it; the *parent* of a vertex $v \in \mathcal{T}_n$, if $n > 0$, is the vertex $v' \in \mathcal{T}_{n-1}$ neighboring it.

9.2.3. The multiplicative monoid $\{\alpha \in R : (\text{nr}(\alpha), 2) = 1\}$ acts in an evident way $((\alpha, L) \mapsto \alpha L)$ on \mathcal{T} by isometries that fix the origin, hence stabilize each sphere \mathcal{T}_n . The set \mathcal{L}_{N_1, N_2} consists of the ordered pairs of independent vertices $v_1, v_2 \in \mathcal{T}_{N_1}, \mathcal{T}_{N_2}$; for $\alpha \in R$ with $(\text{nr}(\alpha), 2) = 1$, the set of such pairs fixed by α is convex and has cardinality $\text{Fix}_{N_1, N_2}(\alpha)$.

9.2.4. More generally, the multiplicative monoid R acts (by $(\alpha, L) \mapsto \alpha L$) on \mathcal{T} , although not by isometries in general. Say that $\alpha \in R$ *contracts* a vertex L if αL belongs to the convex hull of L and the origin, or equivalently, if $\alpha L \leq L$. For $\alpha \in R$, the set of ordered pairs of independent vertices $v_1, v_2 \in \mathcal{T}_{N_1}, \mathcal{T}_{N_2}$ contracted by α is convex and has cardinality $\text{Fix}_{N_1, N_2}(\alpha)$.

9.3. Some quadratic characters. We introduce a geometric interpretation of the characters $\chi_1, \chi_2, \chi_3 : S^0 \rightarrow \{\pm 1\}$ defined in §3 (see §11.1 for a further Fourier-analytic interpretation). Denote by v_0 the origin in \mathcal{T} and by v_1, v_2, v_3 its children. For $\beta \in S^0$, let $\alpha \in S$ be such that $(\text{nr}(\alpha), 2) = 1$ and $\alpha - \beta \in \mathbb{Z}$. Then $\alpha \equiv 1 \pmod{2}$, so it fixes v_1, v_2, v_3 and hence permutes each of their children. For $i \in \{1, 2, 3\}$, set $\eta_i(\beta) := 1$ if α fixes the children of v_i and $\eta_i(\beta) := -1$ if it swaps them. Define $\chi'_i := \eta_j \eta_k$ where $\{i, j, k\} = \{1, 2, 3\}$.

Lemma 6. *The triples of characters χ_1, χ_2, χ_3 and $\chi'_1, \chi'_2, \chi'_3$ coincide up to a permutation of indices.*

Proof. There is a canonical conjugacy class of ring maps $\rho : R \rightarrow M_2(\mathbb{Z}/4)$, given by the action of R on $E[4]$, under which elements $\beta \in S^0$ have the form

$$\rho(\beta) = \begin{pmatrix} a & 2b \\ 2c & -a \end{pmatrix}.$$

In that optic, $\chi_1(\beta), \chi_2(\beta), \chi_3(\beta) := (-1)^{b+c}, (-1)^{a+c}, (-1)^{a+b}$. Using a basis for \mathbb{Z}_2^2 compatible with ρ , we may assume after relabeling indices that v_1, v_2, v_3 correspond to the respective cyclic subgroups $\frac{1}{2}(1, 1)^t + \mathbb{Z}_2^2, \frac{1}{2}(1, 0)^t + \mathbb{Z}_2^2, \frac{1}{2}(0, 1)^t + \mathbb{Z}_2^2$ of $E[2]$. Let $\beta \in S^0$. Set

$$\alpha := \begin{pmatrix} 1 + 2a & 2b \\ 2c & 1 \end{pmatrix} \in S.$$

Then $(\text{nr}(\alpha), 2) = 1$ and $\alpha - \beta \in \mathbb{Z}$. Using this choice of α , we compute directly that $\eta_1(\beta), \eta_2(\beta), \eta_3(\beta) = (-1)^{a+b+c}, (-1)^c, (-1)^b$; for example, $\eta_1(\beta) = +1$ iff α stabilizes $\frac{1}{4}(1, 1)^t + \mathbb{Z}_2^2$ iff $(\frac{a+b}{2}, \frac{c}{2})^t \in \frac{1}{4}(1, 1)^t + \mathbb{Z}_2^2$ iff $a + b + c$ is even. It follows as required that $\chi_i = \chi'_i$. \square

We assume henceforth that v_1, v_2, v_3 have been ordered so that $\chi_i = \chi'_i$.

9.4. Combinatorial arguments.

Lemma 7. *Let N_1, N_2 be nonnegative integers and $\alpha \in R$.*

- (i) *For $t \in \mathbb{Z}$, one has $\text{Fix}_{N_1, N_2}(t + \alpha) = \text{Fix}_{N_1, N_2}(\alpha)$.*
- (ii) *If $N_1, N_2 \geq 2$, then $\text{Fix}_{N_1, N_2}(2\alpha) = 4 \text{Fix}_{N_1-1, N_2-1}(\alpha)$.*

Proof. (i): Immediate from the definition. (ii): For $N \geq 2$, an element $\alpha \in R$ contracts some $v \in \mathcal{T}^{N-1}$ if and only if 2α contracts both (equivalently, either) of the children of v ; each independent ordered pair contributing to $\text{Fix}_{N_1-1, N_2-1}(\alpha)$ thus corresponds to 2^2 independent ordered pairs contributing to $\text{Fix}_{N_1, N_2}(2\alpha)$, and vice-versa. \square

Lemma 8. *Let $\alpha \in R$ with $(\text{nr}(\alpha), 2) = 1$. Suppose $\text{Fix}_{N_1, N_2}(\alpha) \neq 0$ for some $N_1, N_2 \geq 1$. Then $\alpha \in S$.*

Proof. The hypotheses imply that α fixes at least one pair of independent vertices not equal to the origin, hence fixes their convex hull, hence fixes some pair of children of the origin, hence fixes all three children, i.e., acts trivially on $E[2]$, and so is congruent to a scalar modulo 2, as required. \square

Lemma 9. *Let $\alpha \in R$. Suppose $\text{Fix}_{N_1, N_2}^\#(\alpha) \neq 0$ for some $N_1, N_2 \geq 2$. Then $\alpha \in S$.*

Proof. If $(\text{nr}(\alpha), 2) = 1$, then the conclusion follows from Lemma 8, so suppose $\text{nr}(\alpha) \equiv 0 \pmod{2}$. If $\alpha \in 2R$, the conclusion is clear. It remains to show for α having rank 1 reduction mod 2 that $\text{Fix}_{N_1, N_2}^\#(\alpha) = 0$. The set V of vertices $v \in \mathcal{T}$ contracted by α then contains the origin, is convex, and contains at least one and at most two children of the origin, for else the mod 2 reduction of α would be either invertible or a scalar. If V contains exactly one child of the origin, then it contains no pair of independent vertices in the complement of the origin, and so $\text{Fix}_{N'_1, N'_2}(\alpha) = 0$ for all $N'_1, N'_2 \geq 1$.

It remains to consider the case that V contains exactly two children of the origin, say v_1, v_2 . The matrix of α on $E[2]$ with respect to generators of v_1, v_2 is then diagonal, degenerate, and nonzero. Therefore $\text{tr}(\alpha) \equiv 1 \pmod{2}$; since $\text{nr}(\alpha) \equiv 0 \pmod{2}$, we deduce that $\text{tr}(\alpha)^2 - 4\text{nr}(\alpha)$ is congruent to 1 mod 8 and so is a square in \mathbb{Z}_2^\times . It follows that α acts on $E[2^N]$ for each $N \geq 1$ by a diagonal matrix with distinct entries, and so for each vertex $v \in V$ other than the origin, V contains exactly one of the children of v . There are thus two infinite non-backtracking one-sided paths Z_1, Z_2 , starting from the origin of \mathcal{T} and containing distinct children of the origin, so that $V = Z_1 \cup Z_2$. Thus $\text{Fix}_{N'_1, N'_2}(\alpha) = 2$ for all $N'_1, N'_2 \geq 1$. The claim $\text{Fix}_{N_1, N_2}^\#(\alpha) = 0$ for $N_1, N_2 \geq 2$ follows by inclusion-exclusion. \square

Proof of Proposition 5. Suppose first that $N = 2$. By Lemma 9, one has $\alpha \in S = \mathbb{Z} \oplus S^0$, thus $\alpha = m + \beta$ for some $m \in \mathbb{Z}$, $\beta \in S^0$. By Lemma 7 (i), we reduce (adjusting m as necessary) to the terminologically simpler case in which $\text{nr}(\alpha)$ is odd. We compute separately each term in $\text{Fix}_{2,2}^\#(\alpha)$:

- Since α is a scalar mod 2, it fixes all children v_1, v_2, v_3 of the origin v_0 , so $\text{Fix}_{1,1}(\alpha) = 6$ is the number of distinct ordered pairs of such.
- α fixes either child of v_i iff $\eta_i(\beta) = 1$, in which case it fixes both children and also both of v_j, v_k , so $\text{Fix}_{2,1}(\alpha) = \text{Fix}_{1,2}(\alpha) = 2 \sum_{i=1,2,3} (1 + \eta_i(\beta))$.
- For a distinct ordered pair $i, j \in \{1, 2, 3\}$, α fixes a child of each of v_i and v_j iff $\eta_i(\beta) = \eta_j(\beta) = 1$, in which case it fixes all 4 such pairs of children, hence $\text{Fix}_{2,2}(\alpha) = 2 \sum_{\{i,j\} \subseteq \{1,2,3\}} (1 + \eta_i(\beta))(1 + \eta_j(\beta))$, with the sum taken over unordered pairs.

Thus

$$\text{Fix}_{2,2}^\#(\alpha) = 2 \sum_{\{i,j\} \subseteq \{1,2,3\}} (1 + \eta_i(\beta))(1 + \eta_j(\beta)) - 4 \sum_{i \in \{1,2,3\}} (1 + \eta_i(\beta)) + 6.$$

Simplifying, we obtain $\text{Fix}_{2,2}^\#(\alpha) = 2 \sum_{i=1,2,3} \chi_i(\beta)$, as required.

Suppose now that $N \geq 3$ and that the conclusion holds for smaller values of N . By Lemma 9, $\alpha \in S = \mathbb{Z} + 2R$, and so there is $t \in \mathbb{Z}$ and $\gamma \in R$ for which

$\alpha - t = 2\gamma$. By Lemma 7, we have $\text{Fix}_{N,N}^\#(\alpha) = \text{Fix}_{N,N}^\#(2\gamma) = 4\text{Fix}_{N-1,N-1}^\#(\gamma)$, and so the conclusion follows inductively. \square

9.5. The mean statistics. The following result was promised in §1 (see (6)) to justify interpreting V_N as a variance. We do not use it otherwise.

Proposition 10. *Let $N \geq 2$. Then*

- (i) $|\mathcal{F}_N| = (1 + 2^{-1})(1 - 2^{-1})^2 2^{2N} \frac{23-1}{12}$.
- (ii) $|\mathcal{F}_N|^{-1} \sum_{\varphi \in \mathcal{F}_N} \mu_\varphi = \mu$.

The proof requires a lemma similar to the torsion-freeness of $\Gamma_0(4)/\{\pm 1\}$:

Lemma 11. *Let $N_1, N_2 \geq 0$ with $N_1 + N_2 \geq 2$. Let $\alpha \in R$ with $\text{nr}(\alpha) = 1$. Then $\text{Fix}_{N_1, N_2}(\alpha) = 1_{\alpha=\pm 1} |\mathcal{L}_{N_1, N_2}|$.*

Proof. If $\alpha = \pm 1$, then it fixes every subgroup. Assume otherwise. Then α is a non-scalar unit in a definite quaternionic order, so $\text{nr}(\alpha) = 1$ and $\text{tr}(\alpha) \in \{-1, 0, 1\}$. Suppose $\text{Fix}_{N_1, N_2}(\alpha) \neq 0$, so that $\alpha C_1 = C_1, \alpha C_2 = C_2$ for some $(C_1, C_2) \in \mathcal{L}_{N_1, N_2}$. Consider the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/4\mathbb{Z})$ of α on $E[4]$ with respect to a basis v_1, v_2 for which $\mathbb{Z}v_i \supseteq C_i \cap E[4]$. From $N_1 + N_2 \geq 2$ we obtain $bc \equiv 0 \pmod{4}$, hence from $\text{nr}(\alpha) = 1$ that $ad \equiv 1 \pmod{4}$ and so $\text{tr}(\alpha) \equiv a + d \equiv 2 \pmod{4}$, contradicting that $\text{tr}(\alpha) \in \{-1, 0, 1\}$. \square

We now deduce Proposition 10. Let $\Psi : \mathbf{Y} \rightarrow \mathbb{R}$. Let $g_E \in G$ represent $E \in \mathbf{Y}$. Recall from §2.4 that $\int_{\mathbf{X}} \Psi = \sum_{E \in \mathbf{Y}} \frac{\Psi(E)}{w_E}$. By the $n = 1$ case of (26) and (25) applied to Ψ rather than Ψ_k , we have $\sum_{\mathcal{F}_N} \mu_\varphi(\Psi) = \int_{\mathbf{X}} \Psi(g) \sum_{\Gamma} f(g^{-1}\gamma g) = \sum_{E \in \mathbf{Y}} \frac{\Psi(E)}{w_E} \sum_{\Gamma} \phi^K(g_E^{-1}\gamma g_E)$. If $g_E = 1$, then (33) and Lemma 11 give $\sum_{\Gamma} \phi^K(\gamma) = (1/2) \sum_{\alpha \in R: \text{nr}(\alpha)=1} \text{Fix}_{N,N}^\#(\alpha) = |\mathcal{L}_{N,N}| - |\mathcal{L}_{N-1,N}| - |\mathcal{L}_{N,N-1}| + |\mathcal{L}_{N-1,N-1}| = 2^{2N}(1 + 2^{-1})(1 - 2^{-1})^2$. By the proof of Lemma 11 applied to $R_E := R[1/2] \cap g_E R_2 g_E^{-1}$ rather than R , we obtain the same formula for $\sum_{\Gamma} \phi^K(g_E^{-1}\gamma g_E)$. Since the μ_φ are probability measures, we deduce (ii). By taking $\Psi = 1$ and applying Eichler's mass formula $\sum_{E \in \mathbf{Y}} 1/w_E = \frac{23-1}{12}$ [23], we obtain (i).¹⁸

10. PROOF OF THE ANALYTIC INPUT

10.1. Overview. We now prove (21). Recall from §5 the definition of θ . Recall that $|\theta|^2 \notin L^2$. Fix a modular form Φ on some congruence quotient satisfying

$$\Phi(z) \ll \text{height}(z)^{1/2-\delta} \text{ for some } \delta > 0. \quad (35)$$

Let $N \rightarrow \infty$ be a positive integral parameter. We aim to show that the translates $t(2^{-N})\Phi(z) := \Phi(2^{2N}z)$ satisfy

$$\langle |\theta|^2, t(2^{-N})\Phi \rangle = \langle |\theta|^2, 1 \rangle \langle 1, \Phi \rangle + O_\Phi(N2^{-N}). \quad (36)$$

This gives (21) because h_k, h_l are fixed and cuspidal. The contents of this section have been developed much more generally in our preprint [50]; we retain this section for completeness, noting that we require here only a very special case of the general results of [50]. The reader might profitably consult [50, §3] for a toy version of the argument to follow.

¹⁸One could alternatively apply the Eichler lift and cite known dimension formulas for the space of newforms on $\Gamma_0(2^{2N} \cdot 23)$ [44].

10.1.1. *Remark.* The families \mathcal{F}_N considered in this article consist of automorphic forms with trivial central character. If one instead considers families of forms with central character $\chi = \prod \chi_v : \mathbb{A}^\times / \mathbb{Q}^\times \rightarrow \mathbb{C}^{(1)}$ satisfying $\chi_v(-1) = -1$ for at least one place v , then it turns out that a *cuspidal* elementary theta function takes the place of θ in our argument. The analogue of (36) then follows (most directly with the weaker error $O(N2^{-(1-2\theta)N})$) from the standard L^2 -based estimate (24) and the considerations of this section become unnecessary.

10.2. Since $|\theta|^2$ lives on $\Gamma_0(8)$, we reduce formally (for mild technical convenience) to the case that Φ belongs to the space

$$\mathcal{A}(2^\infty) := \varinjlim_{n \rightarrow \infty} \{ \text{smooth } \Phi : \Gamma(2^n) \backslash \mathbb{H} \rightarrow \mathbb{C} \}.$$

We equip $\mathcal{A}(2^\infty)$ with the normalized Petersson inner product $\langle \cdot, \cdot \rangle$ as in §2.6. We consider the pairing $\langle \Phi_1, \Phi_2 \rangle$ to be defined whenever $|\Phi_1 \Phi_2|$ is integrable.

10.3. Change of polarization and Poisson summation. We begin by developing a regularized spectral decomposition of the function $|\theta|^2 \notin L^2$. The transformations that follow amount to regarding $|\theta|^2$ as the restriction to the first factor of a theta kernel on $\mathrm{SL}_2 \times \mathrm{O}_2$ and decomposing the latter with respect to the action of the second factor O_2 . By definition,

$$|\theta|^2(z) = y^{1/2} \sum_{\substack{m, n \in \mathbb{Z}: \\ \gcd(m, 2) = \gcd(n, 2) = 1}} e((m^2 - n^2)z) \exp(-2\pi(m^2 + n^2)y).$$

We change variables $m, n := (\mu + \nu)/2, (\mu - \nu)/2$ and apply Poisson summation to ν . To keep track of the 2-adic summation conditions and weights that intervene, it will be technically convenient to introduce some 2-adic analysis. Thus, define the Schwartz–Bruhat function $\phi \in \mathcal{S}(\mathbb{Q}_2^2)$ by $\phi(m, n) := 1_{\mathbb{Z}_2^\times}(m) 1_{\mathbb{Z}_2^\times}(n)$, denote by $\psi : \mathbb{Q}_2 \rightarrow \mathbb{C}^{(1)}$ the standard character for which $e(x)\psi(x) = 1$ for $x \in \mathbb{Z}[1/2]$, and introduce the partial Fourier transform $\mathcal{F}\phi \in \mathcal{S}(\mathbb{Q}_2^2)$ by

$$\mathcal{F}\phi(y_1, y_2) := \int_{t \in \mathbb{Q}_2} \phi\left(\frac{y_1 + t}{2}, \frac{y_1 - t}{2}\right) \psi(y_2 t) dt$$

where dt assigns unit volume to \mathbb{Z}_2 . By Poisson summation for $\mathbb{Z}[1/2] \hookrightarrow \mathbb{R} \times \mathbb{Q}_2$,

$$\begin{aligned} |\theta|^2(z) &= y^{1/2} \sum_{m, n \in \mathbb{Z}[1/2]} \phi(m, n) e((m^2 - n^2)x) \exp(-2\pi(m^2 + n^2)y) \\ &= y^{1/2} \sum_{\mu, \nu \in \mathbb{Z}[1/2]} \phi\left(\frac{\mu + \nu}{2}, \frac{\mu - \nu}{2}\right) e(\mu\nu x) \exp(-\pi(\mu^2 + \nu^2)y) \\ &= \sum_{\mu, \nu \in \mathbb{Z}[1/2]} \mathcal{F}\phi(\mu, \nu) \exp(-\pi((\mu x + \nu)^2/y + \mu^2 y)) \\ &= \sum_{\mu, \nu \in \mathbb{Z}[1/2]} \mathcal{F}\phi(\mu, \nu) \exp(-\pi|\mu z + \nu|^2/y). \end{aligned}$$

10.4. Some notation. Introduce the general notation

$$n(b) := \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \quad t(a) := \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \quad e_2 := (0, 1).$$

Denote by N_2, T_2, B_2 the subgroups of $\mathrm{SL}_2(\mathbb{Q}_2)$ consisting of elements of the respective forms $n(*), t(*), n(*)t(*)$. Denote by $\tilde{\Gamma}_\infty < \mathrm{SL}_2(\mathbb{Z}[1/2])$ and $\Gamma_\infty < \mathrm{SL}_2(\mathbb{Z})$ the intersections with the upper-triangular Borel subgroups; the corresponding quotients are $\mathbb{P}^1(\mathbb{Z}[1/2]) \cong \mathbb{P}^1(\mathbb{Z})$.

10.5. Folding up, Mellin expansion. Each $(\mu, \nu) \in \mathbb{Z}[1/2]^\times - \{(0, 0)\}$ is uniquely of the form $\lambda e_2 g$ for some $g \in \tilde{\Gamma}_\infty \setminus \mathrm{SL}_2(\mathbb{Z}[1/2])$ and $\lambda \in \mathbb{Z}[1/2] - \{0\}$. Since then $|\mu z + \nu|^2/y = \lambda^2/\mathrm{Im}(gz)$, we obtain

$$|\theta|^2(z) = \mathcal{F}\phi(0, 0) + \sum_{\gamma \in \tilde{\Gamma}_\infty \setminus \mathrm{SL}_2(\mathbb{Z}[1/2])} F(\gamma z, \gamma)$$

where $F : \mathbb{H} \times \mathrm{SL}_2(\mathbb{Q}_2) \rightarrow \mathbb{C}$ is given by

$$F(z, g) := \sum_{\lambda \in \mathbb{Z}[1/2] - \{0\}} \exp(-\pi \lambda^2/\mathrm{Im}(z)) F\phi(\lambda e_2 g).$$

The second term is an incomplete Eisenstein series which we now study by Mellin expansion. For fixed $z \in \mathbb{H}$ and $g \in \mathrm{SL}_2(\mathbb{Q}_2)$, the function of the variable $a = (a_\infty, a_2) \in \mathbb{R}^\times \times \mathbb{Q}_2^\times$ given by $a \mapsto F(a_\infty^2 z, t(a_2)g)$ is left-invariant by the diagonal embedding of $\mathbb{Z}[1/2]^\times$, right-invariant by $\{\pm 1\} \hookrightarrow \mathbb{R}_+^\times$, decays rapidly as $|a| := |a_\infty|_\infty |a_2|_2$ tends to zero, and is $O(|a|^{O(1)})$ as $|a| \rightarrow \infty$. It thus admits a Mellin expansion indexed by

$$\mathfrak{X} := \left\{ \chi = (\chi_\infty, \chi_2) \in \mathrm{Hom}(\mathbb{R}^\times \times \mathbb{Q}_2^\times, \mathbb{C}^\times) \left| \begin{array}{l} \chi|_{\mathbb{Z}[1/2]^\times} = 1, \\ \chi_\infty(-1) = 1 \end{array} \right. \right\}$$

and given for large enough $c > 2$ by $F(z, g) = \int_{\chi \in \mathfrak{X} : \mathrm{Re}(\chi) = c} F_\chi(z, g) d\chi$ with

$$F_\chi(z, g) := \int_{a \in \mathbb{Z}[1/2]^\times \setminus (\mathbb{R}^\times \times \mathbb{Q}_2^\times)} \chi^{-1}(a) F(a_\infty^2 z, t(a_2)g) d^\times a.$$

Here we normalize measures by taking on $\mathbb{Z}[1/2]^\times \setminus (\mathbb{R}^\times \times \mathbb{Q}_2^\times)$ the quotient of the product of the standard Haar measures on \mathbb{R}^\times and \mathbb{Q}_2^\times assigning unit volume to $(1, e)$ and \mathbb{Z}_2^\times , respectively, and on each $\{\chi \in \mathfrak{X} : \mathrm{Re}(\chi) = c\}$ the dual measure $d\chi$. Concretely, \mathfrak{X} identifies with the set of pairs $\chi \leftrightarrow (s_\chi, \omega_\chi)$, where $s_\chi \in \mathbb{C}$ and $\omega_\chi : \mathbb{Z}_2^\times \rightarrow \mathbb{C}^{(1)}$ is a character satisfying $\omega_\chi(-1) = 1$. This identification is determined by requiring that $\chi_\infty(y) = |y|^{s_\chi}$ for $y \in \mathbb{R}^\times$ and $\chi_2(2^n u) = 2^{-ns_\chi} \omega_\chi(u)$ for $n \in \mathbb{Z}$ and $u \in \mathbb{Z}_2^\times$. The real part, analytic conductor and dual measure are given in these coordinates by $\mathrm{Re}(\chi) = \mathrm{Re}(s_\chi)$, $C(\chi) = (1 + |s_\chi|)C(\omega_\chi)$ and $d\chi = \frac{ds_\chi}{2\pi i} d\omega_\chi$, where $d\omega_\chi$ denotes counting measure. We remark that only those χ with $C(\omega_\chi) \leq 2^4$ (say) are required in our argument.

10.6. L-functions. For $\chi \in \mathfrak{X}$, write $\chi_p(p) := \chi_\infty(p)^{-1} \chi_2(p)^{-1}$ for $p > 2$ and define

$$\Lambda(\chi, s) := L(\chi_\infty, s) L(\chi_2, s) \prod_{p>2} (1 - \chi_p(p) p^{-s})^{-1}$$

for $\operatorname{Re}(s)$ large enough and in general by meromorphic continuation. Thus $\Lambda(\chi, s)$ is the completed L -function obtained by regarding χ as a Hecke character unramified outside $\{\infty, 2\}$. For orientation, we record that the element $|\cdot|^s \in \mathfrak{X}$ with components $(|\cdot|_\infty^s, |\cdot|_2^s)$ has $\Lambda(|\cdot|^s, 0) = \xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$.

10.7. Unfolding. We now explicate F_χ further by opening the sum defining F . The positive odd integers give representatives for the $\mathbb{Z}[1/2]^\times$ -orbits on $\mathbb{Z}[1/2] - \{0\}$, so $F_\chi(z, g)$ unfolds to

$$\sum_{\substack{\lambda \geq 1: \\ \gcd(\lambda, 2) = 1}} \chi^{-1}(\lambda) \int_{a \in \mathbb{R}^\times \times \mathbb{Q}_2^\times} \chi^{-1}(a) \exp\left(\frac{-\pi}{\operatorname{Im}(z)a_\infty^2}\right) \mathcal{F}\phi(a_2^{-1}e_2g) d^\times a.$$

By evaluating the Dirichlet series and local Tate integrals, we obtain $F_\chi(z, g) = \Lambda(\chi, 0)f_\chi(z, g)$ with

$$f_\chi(z, g) := \frac{\chi_\infty(\operatorname{Im}(z)^{1/2})}{L(\chi_2, 0)} \int_{a \in \mathbb{Q}_2^\times} \chi_2(a) \mathcal{F}\phi(ae_2g) d^\times a.$$

There exists an open subgroup $U \leq \operatorname{SL}_2(\mathbb{Z}_2)$, independent of everything, so that $f_\chi(z, gu) = f_\chi(z, g)$ for all $u \in U$. By the theory of local Tate integrals or direct evaluation, one has $\|f_\chi\| \ll 1$ for all $\chi \in \mathfrak{X}$ with $-10c \leq \operatorname{Re}(\chi) \leq 10c$, say.

10.8. Induced representations, Eisenstein series. The function f_χ belongs to the space

$$\mathcal{I}(\chi) := \left\{ f : \mathbb{H} \times \operatorname{SL}_2(\mathbb{Q}_2) \rightarrow \mathbb{C} \left| \begin{array}{l} f(z, n(b)t(a)g) = \chi(\operatorname{Im}(z)^{1/2}, a)f(i, g) \\ \text{for } z, a, b, g \in \mathbb{H}, \mathbb{Q}_2^\times, \mathbb{Q}_2, \operatorname{SL}_2(\mathbb{Q}_2), \\ f \text{ is smooth} \end{array} \right. \right\}$$

which arises naturally as the $\operatorname{SO}(2)$ -fixed subspace of the representation of $\operatorname{SL}_2(\mathbb{R} \times \mathbb{Q}_2)$ induced by χ (without normalization). The space $\mathcal{I}(\chi)$ is stable for the Laplacian, which acts by the scalar $\frac{1}{2}s_\chi(\frac{1}{2}s_\chi - 1)$, and under right translation by $\operatorname{SL}_2(\mathbb{Q}_2)$. Denote by $\operatorname{Eis} : \mathcal{I}(\chi) \rightarrow \mathcal{A}(2^\infty)$ the standard intertwiner given by

$$\operatorname{Eis}(f)(z) := \sum_{\gamma \in \tilde{\Gamma}_\infty \setminus \operatorname{SL}_2(\mathbb{Z}[1/2])} f(\gamma z, \gamma) = \sum_{\gamma \in \Gamma_\infty \setminus \operatorname{SL}_2(\mathbb{Z})} f(\gamma z, \gamma)$$

for $\operatorname{Re}(\chi) > 2$ and in general by meromorphic continuation along flat sections. This assignment is equivariant for Δ and $\operatorname{SL}_2(\mathbb{Q}_2)$; the latter acts on $\mathcal{A}(2^\infty)$ by $g\Phi(z) := \Phi(\alpha^{-1}z)$ for $\alpha \in \operatorname{SL}_2(\mathbb{Z}[1/2])$ taken 2-adically close enough to $g \in \operatorname{SL}_2(\mathbb{Q}_2)$. Set $\operatorname{Eis}^*(f) := \Lambda(\chi, 0) \operatorname{Eis}(f)$; it is defined for all $\chi \neq |\cdot|^2$ with $\operatorname{Re}(\chi) \geq 1$, noting that the pole of $\Lambda(\chi, 0)$ at $\chi = |\cdot|^1$ cancels the simple zero of Eis at that parameter. Then

$$|\theta|^2(z) = \mathcal{F}\phi(0, 0) + \int_{\chi \in \mathfrak{X} : \operatorname{Re}(\chi) = c} \operatorname{Eis}^*(f_\chi)(z) d\chi.$$

10.9. Example. Suppose $\chi = |\cdot|^s$ and $f_\chi^0 \in \mathcal{I}(\chi)$ is defined by requiring that $f_\chi^0(i, k) = 1$ for all $k \in \operatorname{SL}_2(\mathbb{Z}_2)$. Then $\operatorname{Eis}(f_\chi^0)(z) = E_{s/2}(z)$ and $\operatorname{Eis}^*(f_\chi^0)(z) = \xi(s)E_{s/2}(z)$ with $E_s(z) = y^s + \dots$ as in §1.

10.10. Measures. Equip N_2, T_2 with the measures transported by the isomorphisms $n : \mathbb{Q}_2 \cong N_2, t : \mathbb{Q}_2^\times \cong T_2$, B_2 with the left Haar compatible with $B_2/N_2 \cong T_2$ and the chosen measures on N_2, T_2 , and $\mathrm{SL}_2(\mathbb{Q}_2)$ with the Haar inducing a quotient Haar for which $N_2 \backslash \mathrm{SL}_2(\mathbb{Q}_2) \ni g \mapsto e_2 g \in \mathbb{Q}_2^\times$ is measure-preserving. Equip $B_2 \backslash \mathrm{SL}_2(\mathbb{Q}_2)$ with the quotient Haar.

10.11. Contour shift. We now shift to $\mathfrak{X}_u := \{\chi \in \mathfrak{X} : \mathrm{Re}(\chi) = 1\}$,¹⁹ passing a pole at $\chi = |\cdot|^2$ of residue $\int_{g \in N_2 \backslash \mathrm{SL}_2(\mathbb{Q}_2)} \mathcal{F}\phi(e_2, g) = \int_{\mathbb{Q}_2^\times} \mathcal{F}\phi$; see §10.12 below for details. Because the function $1_{\mathbb{Z}_2^\times}$ is even, we see by Fourier inversion that $\mathcal{F}\phi(0, 0) = \int_{\mathbb{Q}_2^\times} \mathcal{F}\phi = (1/2) \int_{x \in \mathbb{Q}_2} \phi(x, x) = 1/4$. Thus $|\theta|^2(z) = 1/2 + \int_{\chi \in \mathfrak{X}_u} \mathrm{Eis}^*(f_\chi)(z) d\chi$. We integrate both sides against the constant function 1 and the function Φ from §10.1, using standard growth estimates on Eisenstein series (see §10.13 below) and the boundedness of f_χ (see §10.7) and the rapid decay of $\Lambda(\chi, 0)$ for bounded $C(\omega_\chi)$ to justify changing the order of integration. We obtain

$$\|\theta\|^2 = 1/2 \quad (37)$$

and

$$|\theta|^2(z) = \langle |\theta|^2, 1 \rangle + \int_{\chi \in \mathfrak{X}_u} \mathrm{Eis}^*(f_\chi)(z) d\chi \quad (38)$$

and

$$\langle |\theta|^2, t(2^{-N})\Phi \rangle = \langle |\theta|^2, 1 \rangle \langle 1, \Phi \rangle + \int_{\chi \in \mathfrak{X}_u} \langle \mathrm{Eis}^*(f_\chi), t(2^{-N})\Phi \rangle d\chi. \quad (39)$$

To complete the proof of (36) it suffices now to show for $A = 1.01$, so that $\int_{\mathfrak{X}_u} C(\chi)^{-A} d\chi < \infty$, that

$$\langle \mathrm{Eis}^*(f_\chi), t(2^{-N})\Phi \rangle \ll_\Phi N 2^{-N} C(\chi)^{-A}. \quad (40)$$

This follows from standard growth and matrix coefficients bounds; see §10.18 for details.

10.12. Residues. When $\chi = |\cdot|^2$, so that each $f \in \mathcal{I}(\chi)$ transforms on the left under the modulus character of B_2 , the representation $\mathcal{I}(\chi)$ is reducible: there is an $\mathrm{SL}_2(\mathbb{Q}_2)$ -invariant map $\mathcal{R} : \mathcal{I}(\chi) \rightarrow \mathbb{C}$ given by $\mathcal{R}(f) := \zeta_2(2) \int_{g \in B_2 \backslash \mathrm{SL}_2(\mathbb{Q}_2)} f(i, g)$, which we have normalized so that $\mathcal{R}(f_\chi^0) = 1$ for f_χ^0 as in §10.9. Let $f_\chi \in \mathcal{I}(\chi)$ vary in a bounded holomorphic family in the strip $2 - \varepsilon < \mathrm{Re}(\chi) < 2 + \varepsilon$, with f_χ invariant by some open $U \leq \mathrm{SL}_2(\mathbb{Q}_2)$ independent of χ ; these assumptions hold in the context of §10.7–§10.11. Then $\mathrm{Eis}(f_\chi)$ is holomorphic away from a simple pole at $\chi = |\cdot|^2$ with residue described by

$$\int_{\chi \in \mathfrak{X} : \mathrm{Re}(\chi) = 2 + \varepsilon} \mathrm{Eis}^*(f_\chi) d\chi - \int_{\chi \in \mathfrak{X} : \mathrm{Re}(\chi) = 2 - \varepsilon} \mathrm{Eis}^*(f_\chi) d\chi = \mathcal{R}(f_{|\cdot|^2}). \quad (41)$$

For example, for $f_\chi = f_\chi^0$, this says that

$$\mathrm{res}_{s \rightarrow 2} \xi(s) E_{s/2}(z) = \mathrm{res}_{s \rightarrow 1} 2\xi(2s) E_s(z) = 1,$$

as is well-known. The general case follows either by noting that both sides of (41) may be interpreted as defining elements of the one-dimensional space of equivariant functionals $\mathcal{I}(|\cdot|^2) \rightarrow \mathbb{C}$ or by applying the general treatment of [21] to SL_2 instead of PGL_2 .

¹⁹The unitary axis is at 1 rather than 1/2 because we are working with SL_2 rather than PGL_2 Eisenstein series, the former being more natural for our purposes.

10.13. Growth bounds. Define $\text{height} : \mathbb{H} \rightarrow \mathbb{C}$ by $\text{height}(z) := \sup_{\gamma \in \text{SL}_2(\mathbb{Z})} \text{Im}(\gamma z)$. It descends to $\text{height} : \Gamma(2^n) \backslash \mathbb{H} \rightarrow \mathbb{C}$. Let U be an open subgroup of $\text{SL}_2(\mathbb{Q}_2)$. Let $\chi \in \mathfrak{X}$ with $\text{Re}(\chi) \geq 1$ and $\text{Re}(\chi) \ll 1$. Let $f \in \mathcal{I}(\chi)$ be U -invariant. Then

$$\text{Eis}^*(f)(z) \ll_{U,A} C(\chi)^{-A} \text{height}(z)^{\text{Re}(\chi)/2} \log(3 + \text{height}(z)) \|f\|. \quad (42)$$

Proof. We may assume $f \neq 0$. Then the U -invariance of f implies $C(\chi_2) \ll_U 1$, hence that $\Lambda(\chi, 0) \ll_A C(\chi)^{-A}$, so it suffices to show that $\text{Eis}^*(f)(z) \ll_U C(\chi)^{O(1)} \text{height}(z)^{\text{Re}(\chi)/2} \log(3 + \text{height}(z)) \|f\|$. For this, we estimate Fourier coefficients on a Siegel domain as in [45, (4.12)] and [45, (3.23)]. \square

10.14. Projections. For $\chi \in \mathfrak{X}^u$, the norm

$$\mathcal{I}(\chi) \ni f \mapsto \|f\|^2 := \int_{g \in B_2 \backslash \text{SL}_2(\mathbb{Q}_2)} |f(i, g)|^2$$

is $\text{SL}_2(\mathbb{Q}_2)$ -invariant. By duality, for $\Phi \in \mathcal{A}(2^\infty)$ satisfying (35) there is a unique $\Phi_\chi \in \mathcal{I}(\chi)$ so that $\langle \text{Eis}(f), \Phi \rangle = \langle f, \Phi_\chi \rangle$ (the first inner product taken in $\mathcal{A}(2^\infty)$, the second in $\mathcal{I}(\chi)$). The maps $\Phi \mapsto \Phi_\chi$ are linear and equivariant for Δ and $\text{SL}_2(\mathbb{Q}_2)$.

10.15. Plancherel theorem. We record for the sake of orientation that if Φ is square-integrable, then $\|\Phi\|^2 = \|\Phi_{\text{disc}}\|^2 + (1/2) \int_{\mathfrak{X}^u} \|\Phi_\chi\|^2 d\chi$, where Φ_{disc} is the orthogonal projection onto the discrete part of $\mathcal{A}(2^\infty)$ spanned by constants and cusp forms.

10.16. Bounds for projections. For $\Phi \in \mathcal{A}(2^\infty)$ satisfying (35) and $A > 0$, we claim that $\Lambda(\chi, 0) \|\Phi_\chi\| \ll_{\Phi,A} C(\chi)^{-A}$ for all $\chi \in \mathfrak{X}^u$. To see this, let U be an open subgroup of $\text{SL}_2(\mathbb{Q}_2)$ that fixes Φ . Then $f := \Phi_\chi \in \mathcal{I}(\chi)$ is U -invariant. By estimating the integral of $|\text{Eis}^*(f)(z) \Phi(z)|$ over a Siegel domain using §10.13, it follows that

$$\Lambda(\chi, 0) \langle f, \Phi_\chi \rangle = \langle \text{Eis}^*(f), \Phi \rangle \ll_{\Phi,A} C(\chi)^{-A} \|f\|.$$

Cancelling common factors of $\|\Phi_\chi\|$ (if nonzero) from $\langle f, \Phi_\chi \rangle = \|\Phi_\chi\|^2$ and $\|f\| = \|\Phi_\chi\|$, we conclude.

10.17. Bounds for matrix coefficients. Let $\chi \in \mathfrak{X}^u$, let U be an open subgroup of $\text{SL}_2(\mathbb{Z}_2)$, let $f_1, f_2 \in \mathcal{I}(\chi)$ be U -invariant, and let $n \in \mathbb{Z}$. By explicating [12], we obtain²⁰

$$|\langle f_1, t(2^n) f_2 \rangle| \leq [\text{SL}_2(\mathbb{Z}_2) : U] (2|n| + 1) 2^{-|n|} \|f_1\| \|f_2\|.$$

10.18. Completion. We deduce the remaining estimate (40) from §10.17 and §10.16:

$$\begin{aligned} \langle \text{Eis}^*(f_\chi), t(2^{-N}) \Phi \rangle &= \Lambda(\chi, 0) \langle f_\chi, t(2^{-N}) \Phi_\chi \rangle \\ &\ll_\Phi N 2^{-N} \|f_\chi\| \Lambda(\chi, 0) \|\Phi_\chi\| \\ &\ll_{\Phi,A} N 2^{-N} C(\chi)^{-A}. \end{aligned}$$

²⁰This estimate may be understood as a general form of the well-known Ramanujan-type bound for the 2^{2n} th Hecke eigenvalues of unitary Eisenstein series on $\text{SL}_2(\mathbb{Z})$.

10.19. Refinements. We discuss heuristically some possible refinements of the above analysis (cf. §6.5), leaving an actual implementation to the interested reader. Recall that $(f_\chi)_\chi$ and Φ are independent of N and U -invariant for some fixed open $U \leq \mathrm{SL}_2(\mathbb{Z}_2)$. This subgroup U is *not* the full maximal compact subgroup $\mathrm{SL}_2(\mathbb{Z}_2)$, but let us pretend for the sake of illustration that it were. Then f_χ and Φ_χ “are” spherical vectors. For $\chi \in \mathfrak{X}^u$, it follows that $f_\chi = 0$ unless $\chi = |\cdot|^{1+2it}$ for some $t \in \mathbb{R}$, in which case we may write $\langle \mathrm{Eis}^*(f_\chi), \Phi_\chi \rangle =: H(t)$ for some Schwartz function $H : \mathbb{R} \rightarrow \mathbb{C}$. By the Macdonald formula for the spherical matrix coefficients of $\mathcal{I}(\chi)$, the inner product $\langle \mathrm{Eis}(f_\chi), t(2^{-N})\Phi \rangle$ may be written $2^{-N}H(t)(2^{Nit} + 2^{(N-2)it} + \dots + 2^{-Nit})$ plus a similar term involving $N-2$ in place of N . Thus $\langle |\theta|^2, t(2^{-N})\Phi \rangle - \langle |\theta|^2, 1 \rangle \langle 1, \Phi \rangle$ is morally

$$2^{-N} \int_{t \in \mathbb{R}} H(t)(2^{Nit} + 2^{(N-2)it} + \dots + 2^{-Nit}) dt = 2^{-N} \sum_{\substack{-N \leq n \leq N: \\ n \equiv N(2)}} \hat{H}(n)$$

for the normalized Fourier transform $\hat{H}(\xi) := \int_{t \in \mathbb{R}} H(t) 2^{i\xi t} dt$. Since \hat{H} decays rapidly, the error is thus $O(2^{-N})$. Similar arguments should apply non-heuristically.

11. CLASSICAL CORRELATIONS

In this section we record some local calculations relevant for specializing the Rallis inner product formula. The main result is Proposition 14. Our proof uses Fourier analysis as in §8, which seems more naturally suited to the task at hand; it should also be possible to argue geometrically as in §9. We retain throughout this section the notation and setup of §9.

11.1. Fourier-analytic interpretation of some quadratic characters. Recall the definition of χ_i from §3. We record here an equivalent Fourier-analytic definition. Recall $\rho : R \rightarrow M_2(\mathbb{Z}/4)$ from the proof of Lemma 6. Recall that (up to permutation) $\chi_1(\alpha), \chi_2(\alpha), \chi_3(\alpha) = (-1)^{b+c}, (-1)^{a+c}, (-1)^{a+b}$. Let e_1, e_2, e_3 be the matrices

$$e_1 := \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad e_2 := \begin{pmatrix} 1 & 1 \\ & -1 \end{pmatrix}, \quad e_3 := \begin{pmatrix} 1 & \\ 1 & -1 \end{pmatrix}.$$

Their reductions modulo 2 form a conjugacy class in $\mathrm{GL}_2(\mathbb{F}_2)$. For matrices A, A_1, A_2 over any commutative ring, denote by $A \mapsto A^\iota$ the main anti-involution (thus $AA^\iota = \det(A)$) and by $\langle A_1, A_2 \rangle := \mathrm{tr}(A_1 A_2^\iota)$ the trace pairing. For $\alpha \in S^0$, the pairings $\langle \rho(\alpha), e_i/2 \rangle \in \mathbb{Z}/2$ are then well-defined, and one has

$$\chi_i(\alpha) = (-1)^{\langle \rho(\alpha), e_i/2 \rangle}. \quad (43)$$

11.2. Fourier transform and adjoint action. Let $B := R \otimes_{\mathbb{Z}} \mathbb{Q}$ be the quaternion algebra generated by R . Introduce the subscript 2 to denote “2-adic completion,” so that $B_2 = B \otimes_{\mathbb{Q}} \mathbb{Q}_2$, $R_2 = R \otimes_{\mathbb{Z}} \mathbb{Z}_2$, $R_2^0 = R^0 \otimes_{\mathbb{Z}} \mathbb{Z}_2 = \{\alpha \in R_2 : \mathrm{tr}(\alpha) = 0\}$, and so on. Fix an identification $B_2 = M_2(\mathbb{Q}_2)$ under which R_2 identifies with $M_2(\mathbb{Z}_2)$. The characters $\chi_i : S^0 \rightarrow \{\pm 1\}$ extend by continuity to S_2^0 . Extending them further by zero, we obtain Schwartz–Bruhat functions $\chi_i : B_2^0 \rightarrow \mathbb{C}$ as in §3. For notational clarity, set $\mathfrak{o} := \mathbb{Z}_2, \mathfrak{p} := 2\mathbb{Z}_2$. Fix an unramified character

$\psi : \mathbb{Q}_2 \rightarrow \mathbb{C}^{(1)}$, i.e., one for which $\{x \in \mathbb{Q}_2 : \psi(x\mathfrak{o}) \subseteq \{1\}\} = \mathfrak{o}$. Equip B_2 with the measure $d\alpha$ with the property that the Fourier transform $\mathfrak{F} \circ \mathcal{S}(B_2^0)$ defined by

$$\mathfrak{F}\phi(\alpha) := \int_{\alpha' \in B_2} \phi(\alpha') \psi(\langle \alpha, \alpha' \rangle) d\alpha', \quad (44)$$

with $\langle \alpha, \alpha' \rangle$ the trace pairing as in §9.3, satisfies $\mathfrak{F}\mathfrak{F}\phi(\alpha) = \phi(-\alpha)$. Define an inner product $\langle \cdot, \cdot \rangle_{L^2(B_2)}$ with respect to $d\alpha$. Set $G := \mathrm{PGL}_2(\mathbb{Q}_2) = B_2^\times / \mathbb{Q}_2^\times$, $K := \mathrm{PGL}_2(\mathbb{Z}_2) = R_2^\times / \mathbb{Z}_2^\times$. For $X = B_2^0$ or $X = B_2$, the group G acts on X , hence on $\mathcal{S}(X)$, by the adjoint action: for $g, \beta, \phi \in G, X, \mathcal{S}(X)$

$$\mathrm{Ad}(g)\beta := g\beta g^{-1}, \quad \mathrm{Ad}(g)\phi(\beta) := \phi(\mathrm{Ad}(g)^{-1}\beta). \quad (45)$$

Recall the Cartan decomposition $G = \bigsqcup_{n \in \mathbb{Z}_{\geq 0}} Ka(2^n)K$ with $a(y) := \mathrm{diag}(y, 1)$. For $g \in G$, denote by $\mathbf{n}(g)$ the integer $n \in \mathbb{Z}_{\geq 0}$ for which $Ka(2^n)K = KgK$.

11.3. Fourier-analytic calculations. Define $\phi, \phi', \phi'' \in \mathcal{S}(B_2), \mathcal{S}(\mathbb{Q}_2), \mathcal{S}(B_2^0)$ by $\phi'(m) := 1_{\mathfrak{o}^\times}(m)$, $\phi''(\beta) := 2^{-3}\kappa_0 \sum_{i=1,2,3} \chi_i(4\beta)$ and by $\phi(m + \beta) := \phi'(m)\phi''(\beta)$ for $m, \beta \in \mathbb{Q}_2, B_2^0$. Thus ϕ'' is as in §3.

Lemma 12. *Let $g \in G$. Set $n := \mathbf{n}(g)$. Then*

$$\langle \mathrm{Ad}(g)\phi, \phi \rangle_{L^2(B_2)} = 2^{-4}\kappa_0^2(1_{n=0}2 + 1)2^{-n}.$$

Proof. Observe first that, since e_1, e_2, e_3 form a conjugacy class mod 2 ,²¹ the function ϕ'' and hence also ϕ is $\mathrm{Ad}(K)$ -invariant. We thereby reduce to the case $g = a(2^n)$. Equip \mathbb{Q}_2 with the Haar measure da assigning volume one to \mathfrak{o} . Equip B_2^0 with the Haar measure $d\beta$ so that for $f \in C_c(B_2^0)$,

$$\int_{\beta \in B_2^0} f(\beta) d\beta := \int_{a,b,c \in \mathbb{Q}_2} f\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right) da db dc.$$

The Haar measure $d\alpha$ on B_2 is then given for $f \in C_c(B_2)$ by

$$\int_{\alpha \in B_2} f(\alpha) d\alpha = \int_{t, \beta \in \mathbb{Q}_2, B_2^0} f(t/2 + \beta) dt d\beta$$

because we may compute that with this normalization, $\mathfrak{F}1_{M_2(\mathbb{Z}_2)} = 1_{M_2(\mathbb{Z}_2)}$. Thus $\langle \mathrm{Ad}(g)\phi, \phi \rangle_{L^2(B_2)} = I' \cdot I''(g)$ with

$$I' := \int_{t \in \mathbb{Q}_2} |\phi'(t/2)|^2 dt = \int_{t \in \mathbb{Q}_2} 1_{\mathfrak{o}^\times}(t/2) dt = 2^{-2}$$

and $I''(g) := \int_{\beta \in B_2^0} \langle \mathrm{Ad}(g)\phi'', \phi'' \rangle$. We must verify that $I''(g) = 2^{-2}3\kappa_0^2$ for $n = 0$ and $I''(g) = 2^{-2-2n}\kappa_0^2$ for $n > 0$. We invoke the Fourier transform \mathcal{F}'' on $\mathcal{S}(B_2^0)$, defined by analogy to (44) using $d\beta$. The measure $d\beta$ is not self-dual for \mathfrak{F}'' but instead satisfies

$$\|f\|^2 = 2^{-1}\|\mathcal{F}''f\|^2 \quad \text{for } f \in L^2(B_2^0) \quad (46)$$

as one verifies by taking (say) $f := 1_{R_2^0}$ and evaluating $\|f\|^2 = 1$, $\mathfrak{F}f = 1_{2^{-1}S_2^0}$, $\|\mathfrak{F}f\|^2 = 2$. By (43), we have $\chi_i(4\beta) = 1_{S^0}(4\beta)\psi(\langle \beta, e_i \rangle)$. We compute

$$\int_{\beta' \in B_2^0} 1_{S^0}(4\beta)\psi(\langle \beta, \beta' \rangle) = 2^4 1_{2R^0}(\beta)$$

²¹alternatively, since the vertices v_1, v_2, v_3 in the definition of the characters χ_i are permuted under tree automorphisms fixing the origin

by expanding the LHS for $\beta = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ to

$$\int_{a', b', c' \in \mathbb{Q}_2} 1_{\mathfrak{o}}(4a') 1_{2\mathfrak{o}}(4b') 1_{2\mathfrak{o}}(4c') \psi(2aa' + bb' + cc') da' db' dc'$$

and evaluating the latter to $2^4 1_{2\mathfrak{o}}(a) 1_{2\mathfrak{o}}(b) 1_{2\mathfrak{o}}(c) = 2^4 1_{2R^0}(\beta)$. Thus

$$\mathcal{F}'' \phi'' = 2^{-3} \kappa_0 \sum_{i=1,2,3} 2^4 1_{e_i+2R_2^0} = 2\kappa_0 \sum_{i=1,2,3} 1_{e_i+2R_2^0}.$$

By (46) and the commutation $\text{Ad}(g) \mathfrak{F}'' \phi'' = \mathfrak{F}'' \text{Ad}(g) \phi''$,

$$I''(g) = 2^{-1} \int_{\beta \in B_2^0} \langle \text{Ad}(g) \mathfrak{F}'' \phi'', \mathfrak{F}'' \phi'' \rangle = 2\kappa_0^2 \sum_{i,j=1,2,3} \langle \text{Ad}(g) 1_{e_i+2R_2^0}, 1_{e_j+2R_2^0} \rangle.$$

For $\beta = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, we see from the formulas

$$\begin{aligned} 1_{e_1+2R_2^0}(\beta) &= 1_{\mathfrak{p}}(a) 1_{\mathfrak{o}^\times}(b) 1_{\mathfrak{o}^\times}(c), \\ 1_{e_2+2R_2^0}(\beta) &= 1_{\mathfrak{o}^\times}(a) 1_{\mathfrak{o}^\times}(b) 1_{\mathfrak{p}}(c), \\ 1_{e_3+2R_2^0}(\beta) &= 1_{\mathfrak{o}^\times}(a) 1_{\mathfrak{p}}(b) 1_{\mathfrak{o}^\times}(c). \end{aligned}$$

that the sets $e_i + 2R_2^0$ ($i = 1, 2, 3$) are disjoint and that their $d\beta$ -volumes are 2^{-3} . If $n = 0$, so that g is the identity, it follows that

$$I''(g) = 2\kappa_0^2 \sum_{i=1,2,3} \|1_{e_i+2R_2^0}\|^2 = 2\kappa_0^2 \cdot 3 \cdot 2^{-3} = 2^{-2} 3\kappa_0^2.$$

Suppose $n \geq 1$. Write $\varpi := 2 \hookrightarrow \mathfrak{p}$. We have $\text{Ad}(g) 1_{e_i+2R^0} = 1_{\text{Ad}(g)(e_i+2R_2^0)}$ and so

$$\begin{aligned} \text{Ad}(g) 1_{e_1+2R_2^0}(\beta) &= 1_{\mathfrak{p}}(a) 1_{\varpi^n \mathfrak{o}^\times}(b) 1_{\varpi^{-n} \mathfrak{o}^\times}(c), \\ \text{Ad}(g) 1_{e_2+2R_2^0}(\beta) &= 1_{\mathfrak{o}^\times}(a) 1_{\varpi^n \mathfrak{o}^\times}(b) 1_{\varpi^{-n} \mathfrak{p}}(c), \\ \text{Ad}(g) 1_{e_3+2R_2^0}(\beta) &= 1_{\mathfrak{o}^\times}(a) 1_{\varpi^n \mathfrak{p}}(b) 1_{\varpi^{-n} \mathfrak{o}^\times}(c). \end{aligned}$$

Thus $\langle \text{Ad}(g) 1_{e_i+2R_2^0}, 1_{e_j+2R_2^0} \rangle = 1_{(i,j)=(2,3)} \int_{a,b,c \in k} 1_{\mathfrak{o}^\times}(a) 1_{\varpi^n \mathfrak{o}^\times}(b) 1_{\mathfrak{o}^\times}(c) da db dc = 1_{(i,j)=(2,3)} 2^{-3-n}$ and so $I''(g) = 2\kappa_0^2 \cdot 2^{-3-n} = 2^{-2-n} \kappa_0^2$, as required. \square

Now equip G with the Haar measure assigning volume one to K . For $k = 1, 2$, denote by $\Xi_k : K \backslash G / K \rightarrow \mathbf{C}$ be the normalized ($\Xi_k(1) = 1$) spherical matrix coefficient attached to Ψ_k (see Lemma 16).

Lemma 13. *The integral $I_0 := \int_{g \in G} \langle \text{Ad}(g) 1_{M_2(\mathbb{Z}_2)}, 1_{M_2(\mathbb{Z}_2)} \rangle_{L^2(B_2)} \Xi_k(g)$ converges absolutely and is given by $I_0 = L_2(\Psi_k, \frac{1}{2}) / \zeta_2(2)$.*

Proof. By direct calculation; see Lemma 16 below. \square

Proposition 14. *For ϕ as in Proposition 12, the integral*

$$I := \int_{g \in G} \langle \text{Ad}(g) \phi, \phi \rangle_{L^2(B_2)} \Xi_k(g)$$

converges absolutely and is given by

$$I = 2^{-4} \kappa_0^2 \left(2 + \frac{L_2(\Psi_k, \frac{1}{2})}{\zeta_2(2)} \right).$$

Proof. We use the Cartan decomposition, writing $G = K \bigsqcup (\bigsqcup_{n>0} Ka(2^n)K)$. For $g = a(2^n)$, one has $\langle \text{Ad}(g)1_{M_2(\mathbb{Z}_2)}, 1_{M_2(\mathbb{Z}_2)} \rangle_{L^2(B_2)} = 2^{-n}$ and thus $\langle \text{Ad}(g)\phi, \phi \rangle = 2^{-4}\kappa_0^2(1_{n=0}2 + \langle \text{Ad}(g)1_{M_2(\mathbb{Z}_2)}, 1_{M_2(\mathbb{Z}_2)} \rangle_{L^2(B_2)})$. Since $\text{vol}(K) = 1$, we conclude by Lemmas 12 and 13. \square

12. RALLIS INNER PRODUCT FORMULA

The main purpose of this section is to prove the identity (23), which is needed to determine the diagonal entries of the limiting quantum variance V_∞ in Theorem 1. The reader who is satisfied with a less precise result in which the diagonal matrix V_∞ is left unspecified may skip this section; on the other hand, the proportionality constants in the quantum variance problem are of basic interest, so it would be disappointing if our method were incapable of determining them.

To prove (23), we specialize the Rallis inner product formula from its general adelic formulation and evaluate some carefully normalized local integrals. The contents of this section are, in principal, straightforward, and should not be confused with representing a primary novelty of this paper. On the other hand, some careful book-keeping is required, and we are not aware of any previous work in which such a specialization was carried out. Indeed, although many classical cases of the Rallis inner product formula are recorded in the literature (see e.g. [9, p285], [10, p55], [40, §6] and its references, [57], [39, p969]), none seem to apply “off-the-shelf” to the theta functions h_1, h_2 considered here. Their methods could likely be adapted, but we have chosen instead to derive a “ready-to-use” specialization of the general adelic form whose application here reduces the proof of (23) to the evaluation of the 2-adic local integral considered in §11. We work in generality here so as not to duplicate effort in future applications of the method.

12.1. Local preliminaries. Let k be a local field of characteristic $\neq 2$. Denote by $\text{Mp}_2(k)$ the metaplectic double cover of $\text{SL}_2(k)$, realized as a set of pairs $(\sigma, \zeta) \in \text{SL}_2(k) \times \{\pm 1\}$ with the multiplication law $(\sigma_1, \zeta_1)(\sigma_2, \zeta_2) = (\sigma_1\sigma_2, \zeta_1\zeta_2c(\sigma_1, \sigma_2))$ for the Kubota cocycle $c : k^\times/k^{\times 2} \times k^\times/k^{\times 2} \rightarrow \{\pm 1\}$; see e.g. [50, §4] for details on this and what follows. As generators for $\text{Mp}_2(k)$ we take for $a, b, \zeta \in k^\times, k, \{\pm 1\}$ the elements

$$n(b) := \left(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, 1 \right), \quad t(a) := \left(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, 1 \right), \quad w := \left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, 1 \right)$$

and $\varepsilon(\zeta) := (1, \zeta)$. Let $\psi : k \rightarrow \mathbb{C}^{(1)}$ be a nontrivial character. Let (V, q) be a quadratic space over k . For $x, y \in V$, set $\langle x, y \rangle := q(x + y) - q(x) - q(y)$. Equip V with the Haar measure dx self-dual for the Fourier transform $\mathcal{F}\phi(y) := \int_V \phi(x)\psi(\langle x, y \rangle) dx$ for ϕ in the Schwartz–Bruhat space $\mathcal{S}(V)$. The Weil representation $\omega_{V, \psi}$ of $\text{Mp}_2(k)$ on $\mathcal{S}(V)$, abbreviated $\omega := \omega_{V, \psi}$ when V, ψ are clear from context, is defined on the generators: there is a quartic character $\chi = \chi_{V, \psi}$ of k^\times and an eighth root of unity $\gamma = \gamma_{V, \psi}$ (see [68, 22]) so that $\omega(n(b))\phi(x) = \psi(bq(x))\phi(x)$, $\omega(t(a))\phi(x) = \chi(a)|a|^{d/2}\phi(ax)$, $\omega(w)\phi = \gamma\mathcal{F}\phi$, and $\omega(\varepsilon(\zeta))\phi = \zeta^{\dim V}\phi$. The assignment $V \mapsto \omega_{V, \psi}$ is compatible with orthogonal direct sums: if $V = V' \oplus V''$, then $\omega_{V, \psi} = \omega_{V', \psi} \otimes \omega_{V'', \psi}$ wrt the dense inclusion $\mathcal{S}(V') \otimes \mathcal{S}(V'') \hookrightarrow \mathcal{S}(V)$.

12.2. Global preliminaries. Now let k be a number field with adèle ring \mathbb{A} . Denote by v a typical place of k . Denote by k_v the completion. For an algebraic group H over k , set $H_v := H(k_v)$. Fix a nontrivial character $\psi : \mathbb{A}/k \rightarrow \mathbb{C}^{(1)}$. Let

(V, q) be a quadratic space over k . The metaplectic group $\mathrm{Mp}_2(\mathbb{A})$ consists of pairs $(\sigma, \zeta) \in \mathrm{SL}_2(\mathbb{A}) \times \{\pm 1\}$ with the law $(\sigma_1, \zeta_1)(\sigma_2, \zeta_2) = (\sigma_1, \sigma_2, \zeta_1 \zeta_2 c(\sigma_1, \sigma_2))$ for the product $c = \prod c_v$ of the local cocycles. The map $\gamma \mapsto (\gamma, 1)$ defines a splitting $\mathrm{SL}_2(k) \hookrightarrow \mathrm{Mp}_2(\mathbb{A})$. Equip $V(\mathbb{A})$ with the product of the local Haar measures on the completions $V(k_v)$. The space $\mathcal{S}(V(\mathbb{A}))$ is a restricted tensor product of the local spaces $\mathcal{S}(V(k_v))$; we assume here that the distinguished elements defining the restricted tensor product are unit vectors for almost all places. The Weil representation $\omega_{\psi, V}$ of $\mathrm{Mp}_2(\mathbb{A})$ on $\mathcal{S}(V(\mathbb{A}))$ is the restricted tensor product of the local Weil representations.

12.3. The main global result. Let B be a quaternion algebra over k . Denote by B^0 the trace zero subgroup and by G the algebraic group with $G(k) = B^\times / k^\times$. Set $[\mathrm{Mp}_2] := \mathrm{SL}_2(k) \backslash \mathrm{Mp}_2(\mathbb{A})$, $[\mathrm{SL}_2] := \mathrm{SL}_2(k) \backslash \mathrm{SL}_2(\mathbb{A})$, $[G] := G(k) \backslash G(\mathbb{A})$. Then $(k, x \mapsto x^2)$, $(B^0, x \mapsto \mathrm{nr}(x))$ and $(B, x \mapsto \mathrm{nr}(x))$ are quadratic spaces with $B = k \oplus B^0$. The above discussion applies, giving Weil representations of $\mathrm{Mp}_2(\mathbb{A})$ on $\mathcal{S}(\mathbb{A}), \mathcal{S}(B_{\mathbb{A}}^0), \mathcal{S}(B_{\mathbb{A}})$ which we denote by ω for notational simplicity. The adjoint action $\mathrm{Ad} : G \rightarrow \mathrm{SO}(B^0)$ defined as in (45) is an isomorphism. Let $\pi \in L^2([G])$ be a cuspidal automorphic representation. Fix a unitary factorization $\pi \cong \otimes \pi_v$. Equip G_v with arbitrary Haar measures; assume only that for non-archimedean v outside some finite set, the maximal compact in G_v has volume one. Equip $G(\mathbb{A})$ with the restricted product measure and $[G]$ the quotient measure.²² For Schwartz functions $\phi', \phi'' \in \mathcal{S}(\mathbb{A}), \mathcal{S}(B^0(\mathbb{A}))$ and $\Psi \in \pi$, define for $\sigma \in [\mathrm{Mp}_2]$ and $g \in [G]$ the adelic theta functions $\theta_{\phi'}(\sigma) := \sum_{m \in k} \omega(\sigma) \phi'(m)$, $\theta_{\phi''}(\sigma, g) := \sum_{\beta \in B^0} \omega(\sigma) \mathrm{Ad}(g) \phi''(\beta)$, $\theta_{\phi', \Psi}(\sigma) := \int_{g \in [G]} \Psi(g) \theta_{\phi'}(\sigma, g)$. Denote simply by \int an integral over $[\mathrm{SL}_2]$ with respect to the Tamagawa measure, which is the probability Haar.

Theorem 15. *For $i = 1, 2$, let $\phi'_i = \otimes \phi'_{i,v} \in \mathcal{S}(\mathbb{A})^{(+)}$, $\phi''_i = \otimes \phi''_{i,v} \in \mathcal{S}(B^0(\mathbb{A}))$ and $\Psi_i = \otimes \Psi_{i,v} \in \pi$. For a sufficiently large finite set S of places,*

$$\int (\theta_{\phi'_1, \Psi_1} \overline{\theta_{\phi'_2, \Psi_2}}) \int (\theta_{\phi'_1} \overline{\theta_{\phi'_2}}) = 2 \frac{L^{(S)}(\pi, \frac{1}{2})}{\zeta_k^{(S)}(2)} \prod_{v \in S} I_v$$

where $I_v := \int_{g \in G_v} \langle \pi_v(g) \Psi_{1,v}, \Psi_{2,v} \rangle_{\pi_v} \langle \mathrm{Ad}(g) \phi_{1,v}, \phi_{2,v} \rangle_{L^2(B_v)}$ with $\phi_{i,v} := \phi'_{i,v} \otimes \phi''_{i,v} \in \mathcal{S}(B_v)$.

Proof. By polarization, it suffices to consider the notationally simpler case $\Psi_{1,v} = \Psi_{2,v} =: \Psi_v$, $\phi'_{1,v} = \phi'_{2,v} =: \phi'_v$, $\phi''_{1,v} = \phi''_{2,v} =: \phi''_v$, $\phi_{1,v} = \phi_{2,v} =: \phi_v$, and similarly without the subscripts “ v .” From the factorization of inner products

$$\langle \mathrm{Ad}(g) \phi_v, \phi_v \rangle = \langle \phi'_v, \phi'_v \rangle \langle \mathrm{Ad}(g) \phi''_v, \phi''_v \rangle$$

according to the orthogonal decomposition $B_v = k_v \oplus B_v^0$, we see that $I_v = I'_v I''_v$, where

$$\begin{aligned} I'_v &:= \langle \phi'_v, \phi'_v \rangle_{L^2(k_v)}, \\ I''_v &:= \int_{g \in G_v} \langle \pi_v(g) \Psi_v, \Psi_v \rangle_{\pi_v} \langle \mathrm{Ad}(g) \phi''_v, \phi''_v \rangle_{L^2(B_v^0)}. \end{aligned}$$

²²It might appear natural to consider here some specific measure on $G(\mathbb{A})$, such as the Tamagawa measure, but the main identities to be considered are self-normalizing for the choice of Haar measure on $G(\mathbb{A})$, so we do not.

It thus suffices to show separately that

$$\int |\theta_{\phi'}|^2 = 2 \prod_{v \in S} I'_v,$$

$$\int |\theta_{\phi'', \Psi}|^2 = \frac{L^{(S)}(\pi, \frac{1}{2})}{\zeta_k^{(S)}(2)} \prod_{v \in S} I''_v.$$

The first identity follows from [50, Thm 4], while the second identity follows from specializing the Rallis inner product given by [20, Thm 6.6]; compare with [58, Prop 2.8 (i)] for a more explicit formulation in the representative case $B = M_2(k)$. \square

12.4. Some local integrals.

Lemma 16 (The unramified case). *Let k be a non-archimedean local field of characteristic zero with ring of integers \mathfrak{o} . Let $\psi : k \rightarrow \mathbb{C}^{(1)}$ be an unramified nontrivial character. Equip $M_2(k)$ with the Haar measure attached to ψ as in §12.1, so that $\text{vol}(M_2(\mathfrak{o})) = 1$. Set $\phi := 1_{M_2(\mathfrak{o})} \in \mathcal{S}(M_2(k))$. Equip $\text{PGL}_2(k)$ with any Haar measure. Let π be an unramified generic irreducible unitary representation of $\text{PGL}_2(k)$ with L^2 -normalized $\text{PGL}_2(\mathfrak{o})$ -invariant vector v and normalized spherical function $\Phi : \text{PGL}_2(k) \rightarrow \mathbb{C}$ given by $\Phi(g) := \langle \pi(g)v, v \rangle$, so that $\Phi(1) = 1$. Then*

$$\int_{g \in \text{PGL}_2(k)} \langle \text{Ad}(g)\phi, \phi \rangle_{L^2(M_2(k))} \Phi(g) = \text{vol}(\text{PGL}_2(\mathfrak{o})) \frac{L(\pi, \frac{1}{2})}{\zeta_k(2)},$$

and the LHS converges absolutely.

Proof. This is implicit in Theorem 15. A direct proof follows from the Cartan decomposition as in §11 and the Macdonald formula as in [11, Thm 4.6.6]; we omit the routine details. \square

Lemma 17 (The non-split case for the trivial representation). *Let k be a local field $\neq \mathbb{C}$ of characteristic zero, $\psi : k \rightarrow \mathbb{C}^{(1)}$ a nontrivial character, and B the non-split quaternion algebra over k . Set $H := PB^\times$; it is compact. Equip B with the Haar measure attached to ψ as in §12.1, and equip H with any Haar measure. If $k = \mathbb{R}$, set $\phi(x) = e^{-2\pi \text{nr}(x)}$ and $q := 1$; if k is non-archimedean, set $\phi := 1_R$ and $q := \#\mathfrak{o}/\mathfrak{p}$, where $R \subset B$ and $\mathfrak{o} \subset k$ are maximal orders and $\mathfrak{p} \subset \mathfrak{o}$ the maximal ideal. Let Φ be the constant function $\Phi(g) := 1$. Let $c \in k^\times$ be such that $\psi(x) = \psi_1(cx)$, where $\psi_1(x) := e^{4\pi i x}$ if $k = \mathbb{R}$ and ψ_1 is unramified if k is non-archimedean. Then*

$$\int_{g \in H} \langle \text{Ad}(g)\phi, \phi \rangle_{L^2(B)} \Phi(g) = \text{vol}(H) q^{-1} |c|^2.$$

Proof. We have $\text{Ad}(g)\phi = \phi$, so the integrand is constant, and the formula to be proved is simply that $\|\phi\|^2 = q^{-1} |c|^2$. Consider first the archimedean case $k = \mathbb{R}$. We have $\|\phi\|^2 = \int_{x \in B} e^{-4\pi \text{nr}(x)} = (|c|^{1/2})^4 \int_{x \in B} e^{-4\pi |c| \text{nr}(x)} = |c|^2 q^{-1} \mathcal{F}\phi_c(0)$, where $\phi_c(x) := \psi(\pm i \text{nr}(x))$ with $\pm 1 := \text{sgn}(c)$; it remains only to verify that $\mathcal{F}\phi_c(0) = 1$. For that, we expand out in terms of Gaussians using coordinates $x = (x_0, x_1, x_2, x_3)$ on B for which $\text{nr}(x) = \sum x_j^2$:

$$\phi_c(x) = \prod e^{-4\pi |c| x_j^2}, \quad \psi(\langle x, y \rangle) = \prod e^{8\pi i |c| x_j y_j}.$$

From this and the known Fourier transform of the Gaussian,²³ we see that $\mathcal{F}\phi_c$ is some multiple of ϕ_c . By the measure normalization, it follows that $\mathcal{F}\phi_c = \phi_c$ and hence $\mathcal{F}\phi_c(0) = 1$, as required. We turn now to the non-archimedean case. Let j be a uniformizer for R . The lattice dual of R with respect to ψ is²⁴ $\{x \in R : \psi(\langle x, R \rangle) = \{1\}\} = c^{-1}j^{-1}R$, so $\mathcal{F}1_R = \gamma 1_{c^{-1}j^{-1}R}$ for some $\gamma > 0$. The volume of $c^{-1}j^{-1}R$ is that of R multiplied by $|\text{nr}(c^{-1}j^{-1})|^2 = |c|^{-4}q^2$, hence $1_R = \mathcal{F}\mathcal{F}1_R = \gamma^2|c|^{-4}q^2 1_R$, giving $\gamma = q^{-1}|c|^2$. Therefore $\|1_R\|^2 = \mathcal{F}1_R(0) = q^{-1}|c|^2$, as required. \square

12.5. Specialization. Suppose now $k := \mathbb{Q}$, that ψ is the standard additive character (thus $\psi = \prod \psi_v$ with $\psi_\infty(x) := e^{2\pi i x}$), that B is a definite quaternion algebra over \mathbb{Q} . Denote by $\text{ram}(B) \ni \infty$ the set of places at which it ramifies. Fix a maximal order $R \subset B$. Let $K := \prod K_v = G_\infty \times K_{\text{fin}}$, $K_{\text{fin}} = \prod_p K_p$ where $K_p = R_p^\times / \mathbb{Z}_p^\times$. Assume that the chosen Haar measure on G_v assigns volume one to each K_v . Then

$$\text{vol}(G_p) = 2 \text{ for all } p \in \text{ram}(B). \quad (47)$$

Let S be a finite set of finite primes containing all $p \in \text{ram}(B)$. Let $\phi' = \otimes \phi'_v \in \mathcal{S}(\mathbb{A})^{(+)}$, $\phi'' = \otimes \phi''_v \in \mathcal{S}(B^0(\mathbb{A}))$ satisfy: for $p \in \text{ram}(B)$ or $p \notin S$,

$$\phi'_\infty(x) = e^{-2\pi x^2}, \phi''_\infty(x) = e^{-2\pi \text{nr}(x)}, \phi'_p = 1_{\mathbb{Z}_p}, \phi''_p = 1_{R_p^0}. \quad (48)$$

Thus ϕ'_v, ϕ''_v is determined except when $v \in S - \text{ram}(B)$. Let $\Psi = \otimes \Psi_v \in \pi = \otimes \pi_v \subseteq L^2([G])$ be a factorized cusp form that is *trivial* at all $v \in \text{ram}(B)$. Assume that the factorization of unitary structures on π is normalized so that $\|\Psi_v\|_{\pi_v} = 1$ for $v \in \text{ram}(B)$ or $v \notin S$. Then $\|\Psi\|_{L^2([G])} = \prod_{p \in S - \text{ram}(B)} \|\Psi_p\|_{\pi_p}$. Set $\tilde{H}_0 := \theta_{\phi'}$, $\tilde{H} := \theta_{\phi'', \Psi}$. Theorem 15 and the results of §12.4 give

$$\int |\tilde{H}_0|^2 \int |\tilde{H}|^2 = 2 \frac{L^{(S)}(\pi, \frac{1}{2})}{\zeta^{(S)}(2)} I_\infty \prod_{p \in S} I_p \quad (49)$$

with $I_\infty = 1/4$ and $I_p = 2/p$ for $p \in \text{ram}(B)$ and I_p as in Theorem 15 for $p \in S - \text{ram}(B)$.

12.6. Unadelization. Let $\Phi : \mathbb{H} \rightarrow \mathbb{C}$ be an automorphic form of weight $k/2$ ($k \in \mathbb{Z}_{\geq 1}$) on $\Gamma(N)$ for some $N \equiv 0 \pmod{4}$. It can be lifted uniquely to an adelic automorphic form $\tilde{\Phi}$ on $[\text{Mp}_2]$ by requiring that $\tilde{\Phi}(n(x)t(y^{1/2})\varepsilon(\zeta)) = \zeta^k \Phi(x + iy)$ for $x, y, \zeta \in \mathbb{R}, \mathbb{R}_+^\times, \{\pm 1\}$; then $\tilde{\Phi}$ transforms under the k th standard character of the standard maximal compact subgroup of $\text{Mp}_2(\mathbb{R})$ and is invariant by the standard splitting of the N th principal congruence subgroup of $\text{SL}_2(\mathbb{A}_f)$ (see e.g. [50, §4.5]). Any automorphic form $\tilde{\Phi}$ on $[\text{Mp}_2]$ with these properties arises in this way from $\Phi(x + iy) := \tilde{\Phi}(n(x)t(y^{1/2}))$, and the bijection $\Phi \leftrightarrow \tilde{\Phi}$ intertwines the inner products defined in §2.6 and §12.3. This discussion applies to \tilde{H}_0, \tilde{H} ; by inspection of the formulas defining the theta lift, one finds that the corresponding classical modular

²³For $y \in \mathbb{R}$, one has $\int_{x \in \mathbb{R}} e^{-4\pi|c|x^2 + 8\pi i|c|xy} dx = (2|c|^{1/2})^{-1} \int_{x \in \mathbb{R}} e^{-\pi x^2 + 2\pi i x(2|c|^{1/2}y)} dx = (2|c|^{1/2})^{-1} e^{-4\pi|c|y^2}$.

²⁴Write $B = E + Ej$, where E/k is the unramified quadratic field extension, $j\alpha j^{-1} = \bar{\alpha}$ for $\alpha \in E$, and $\varpi := j^2$ is a uniformizer of k . Then $R = \mathfrak{o}_E + \mathfrak{o}_E j$ with $\mathfrak{o}_E \subset E$ the maximal order. For $x = x_0 + x_1 j, y = y_0 + y_1 j \in B$, we have $\langle x, y \rangle = \text{tr}(x_0 \bar{y}_0) - \text{tr}(x_1 \bar{y}_1) \varpi$. Thus $x \in R$ satisfies $\psi(\langle x, R \rangle) = \{1\}$ iff $\psi_1(c\langle x, R \rangle) = \{1\}$ iff $\text{tr}(x_0 \mathfrak{o}_E) \subseteq c^{-1} \mathfrak{o}$ and $\text{tr}(x_1 \mathfrak{o}_E) \subseteq c^{-1} \varpi^{-1} \mathfrak{o}$ iff $x \in c^{-1} j^{-1} R$.

forms H_0, H have respective weights $1/2, 3/2, 3/2$ and are given with $\phi'_{\text{fin}} := \otimes \phi'_p$, $\phi''_{\text{fin}} := \otimes \phi''_p$ by

$$H_0(z) = y^{1/4} \sum_{m \in \mathbb{Q}} \phi'_{\text{fin}}(m) q^{m^2}, \quad H(z) = y^{3/4} \int_{g \in [G]} \Psi(g) \sum_{\beta \in B^0} \text{Ad}(g) \phi''_{\text{fin}}(\beta) q^{\text{nr}(\beta)}.$$

By (48), the summations may be restricted to S -integral elements.

12.7. Completion of the calculation. Assume now that B is as in §2, that (48) is satisfied for all $p \notin S := \{2\}$, and that ϕ'_2, ϕ''_2 are as in Lemma 12. These assumptions fully determine ϕ', ϕ'' . They imply that $H_0(z)$ is the Jacobi theta function defined in §5 which we denote here by $\theta_{\text{Jac}}(z) := H_0(z)$. Recall the set $\mathbf{Y} = \Gamma \backslash \text{GL}_2(\mathbb{Q}_2) / \text{GL}_2(\mathbb{Z}_2)$ defined in §1. By strong approximation, one has a bijection $\mathbf{Y} \cong [G]/K$ induced by $\text{GL}_2(\mathbb{Q}_2) \twoheadrightarrow \text{PGL}_2(\mathbb{Q}_2) \cong G(\mathbb{Q}_2) \hookrightarrow G(\mathbb{A})$ (see e.g. [23, §3]). The measure on \mathbf{Y} from §2.4 is compatible with that induced on $[G]/K$ from §12.5 because for $g \in \text{GL}_2(\mathbb{Q}_2)$ with image $[g] \in G(\mathbb{Q}_2)$, one has $(1/2) \# \Gamma \cap g \text{GL}_2(\mathbb{Z}_2) g^{-1} = \# G(\mathbb{Q}) \cap [g]K[g]^{-1}$. Under this bijection, $\Psi_1, \Psi_2 : \mathbf{Y} \rightarrow \mathbb{R}$ identify with factorizable K -invariant vectors (also denoted) $\Psi_k : [G] \rightarrow \mathbb{R}$ in automorphic representations $\pi_1, \pi_2 \subset L^2([G])$ for which $L(\pi_k, s) = L(\Psi_k, s)$. Let $k \in \{1, 2\}$ and set $\Psi = \Psi_k$; one finds using SAGE or otherwise²⁵ that Ψ is trivial at 23 and using §3 that $H(z) = h_k(z)$. We have $\|\Psi_{k,2}\|_{\pi_2}^2 = \|\Psi_k\| = 1$ and hence for $g \in \text{PGL}_2(\mathbb{Q}_2) = B_2^\times / \mathbb{Q}_2^\times$ that $\langle \pi_{k,2}(g) \Psi_{k,2}, \Psi_{k,2} \rangle_{\pi_{k,2}} = \Xi_k(g)$ with Ξ_k as in §11. By (49), it follows that

$$\|\theta_{\text{Jac}}\|^2 \|h_k\|^2 = \int |\tilde{H}_0|^2 \int |\tilde{H}|^2 = 2 \frac{L^{(S)}(\Psi_k, \frac{1}{2})}{\zeta^{(S)}(2)} I_\infty I_{23} I_2$$

with $I_2 = \int_{g \in \text{PGL}_2(\mathbb{Q}_2)} \Xi_k(g) \langle \text{Ad}(g) \phi_2, \phi_2 \rangle_{L^2(B_2)}$ and $I_\infty = 1/4, I_{23} = 2/23$. The measure defining $L^2(B_2)$ is compatible with that in §11, so by Proposition 14 and the definition (§2.8) of κ_0 ,

$$I_2 = 2^{-4} \kappa_0^2 \left(2 + \frac{L_2(\Psi_k, \frac{1}{2})}{\zeta_2(2)} \right) = \kappa_1^{-1} 2^{-6} \zeta_2(1) \zeta_2(2) \left(2 + \frac{L_2(\Psi_k, \frac{1}{2})}{\zeta_2(2)} \right).$$

Recalling the definition (§2.8) of κ_1 , we obtain

$$\|\theta_{\text{Jac}}\|^2 \|h_k\|^2 = 2L^{(S)}(\Psi_k, \frac{1}{2}) \cdot \frac{(4\pi)^2}{4} \cdot \frac{2\zeta_{23}(1)}{23} \cdot 2^{-6} \zeta_2(1)^2 \zeta_2(2) \left(2 + \frac{L_2(\Psi_k, \frac{1}{2})}{\zeta_2(2)} \right).$$

We “simplify” this a bit using that $\zeta_{23}(1) = L_{73}(\Psi_k, \frac{1}{2})$ and factoring $L_2(\Psi_k, \frac{1}{2})$ out of the rightmost expression. Either by inspection or by the SAGE output pasted below, we evaluate the RHS to $L(\Psi_k, \frac{1}{2}) P(\lambda_{\Psi_k}(2))$, as required.

```
lv = 1/(1 - x/sqrt(2) + 1/2); C0 = 2 * (4*pi)^2/4
C2 = 2^(-6)/((1 - 1/2)^2*(1-1/2^2)) * (2/lv + (1-1/2^2))
C23= 2/23; C = C0 * C23 * C2
C
```

$$-1/69 \pi^2 (4 \sqrt{2} x - 15)$$

²⁵e.g., using that the supersingular polynomial $j(j+4)(j-1768)$ splits over \mathbb{F}_{23}

REFERENCES

- [1] A. O. L. Atkin and J. Lehner. Hecke operators on $\Gamma_0(m)$. *Math. Ann.*, 185:134–160, 1970.
- [2] A. Biró. On a generalization of the Selberg trace formula. *Acta Arith.*, 87(4):319–338, 1999.
- [3] András Biró. A relation between triple products of weight 0 and weight $\frac{1}{2}$ cusp forms. *Israel J. Math.*, 182:61–101, 2011.
- [4] S. Böcherer and R. Schulze-Pillot. Vector valued theta series and Waldspurger’s theorem. *Abh. Math. Sem. Univ. Hamburg*, 64:211–233, 1994.
- [5] S. Böcherer and R. Schulze-Pillot. On the central critical value of the triple product L -function. In *Number Theory (Paris, 1993–1994)*, volume 235 of *London Math. Soc. Lecture Note Ser.*, pages 1–46. Cambridge University Press, Cambridge, 1996.
- [6] S. Böcherer and R. Schulze-Pillot. On the central critical value of the triple product L -function. In *Number theory (Paris, 1993–1994)*, volume 235 of *London Math. Soc. Lecture Note Ser.*, pages 1–46. Cambridge Univ. Press, Cambridge, 1996.
- [7] Siegfried Böcherer, Peter Sarnak, and Rainer Schulze-Pillot. Arithmetic and equidistribution of measures on the sphere. *Comm. Math. Phys.*, 242(1-2):67–80, 2003.
- [8] Siegfried Böcherer and Rainer Schulze-Pillot. Siegel modular forms and theta series attached to quaternion algebras. *Nagoya Math. J.*, 121:35–96, 1991.
- [9] Siegfried Böcherer and Rainer Schulze-Pillot. The Dirichlet series of Koecher and Maass and modular forms of weight $\frac{3}{2}$. *Math. Z.*, 209(2):273–287, 1992.
- [10] Siegfried Böcherer and Rainer Schulze-Pillot. Mellin transforms of vector valued theta series attached to quaternion algebras. *Math. Nachr.*, 169:31–57, 1994.
- [11] Daniel Bump. *Automorphic Forms and Representations*, volume 55 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997.
- [12] M. Cowling, U. Haagerup, and R. Howe. Almost L^2 matrix coefficients. *J. Reine Angew. Math.*, 387:97–110, 1988.
- [13] Mladen Dimitrov and Louise Nyssen. Test vectors for trilinear forms when at least one representation is not supercuspidal. *Manuscripta Math.*, 133(3-4):479–504, 2010.
- [14] M. Eichler. The basis problem for modular forms and the traces of the Hecke operators. In *Modular functions of one variable, I (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 75–151. Lecture Notes in Math., Vol. 320. Springer, Berlin, 1973.
- [15] Martin Eichler. Über die Darstellbarkeit von Modulformen durch Thetareihen. *J. Reine Angew. Math.*, 195:156–171 (1956), 1955.
- [16] Matthew Emerton. Supersingular elliptic curves, theta series and weight two modular forms. *J. Amer. Math. Soc.*, 15(3):671–714 (electronic), 2002.
- [17] Brooke Feigon and David Whitehouse. Exact averages of central values of triple product L -functions. *Int. J. Number Theory*, 6(7):1609–1624, 2010.
- [18] Mario Feingold and Asher Peres. Distribution of matrix elements of chaotic systems. *Phys. Rev. A (3)*, 34(1):591–595, 1986.
- [19] Wee Teck Gan, Yannan Qiu, and Shuichiro Takeda. The regularized Siegel-Weil formula (the second term identity) and the Rallis inner product formula. *Invent. Math.*, 198(3):739–831, 2014.
- [20] Wee Teck Gan and Shuichiro Takeda. On the regularized Siegel-Weil formula (the second term identity) and non-vanishing of theta lifts from orthogonal groups. *J. Reine Angew. Math.*, 659:175–244, 2011.
- [21] Stephen Gelbart and Hervé Jacquet. Forms of $GL(2)$ from the analytic point of view. In *Automorphic Forms, Representations and L -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 213–251. Amer. Math. Soc., Providence, R.I., 1979.
- [22] Stephen S. Gelbart. *Weil’s representation and the spectrum of the metaplectic group*. Lecture Notes in Mathematics, Vol. 530. Springer-Verlag, Berlin-New York, 1976.
- [23] Benedict H. Gross. Heights and the special values of L -series. In *Number theory (Montreal, Que., 1985)*, volume 7 of *CMS Conf. Proc.*, pages 115–187. Amer. Math. Soc., Providence, RI, 1987.
- [24] Benedict H. Gross and Stephen S. Kudla. Heights and the central critical values of triple product L -functions. *Compositio Math.*, 81(2):143–209, 1992.
- [25] Michael Harris and Stephen S. Kudla. The central critical value of a triple product L -function. *Ann. of Math. (2)*, 133(3):605–672, 1991.

- [26] Jeffrey Hoffstein and Paul Lockhart. Coefficients of Maass forms and the Siegel zero. *Ann. of Math. (2)*, 140(1):161–181, 1994. With an appendix by Dorian Goldfeld, Hoffstein and Daniel Lieman.
- [27] Y. Hu. Triple product formula and mass equidistribution on modular curves of level N . *ArXiv e-prints*, September 2014.
- [28] Atsushi Ichino. Pullbacks of Saito-Kurokawa lifts. *Invent. Math.*, 162(3):551–647, 2005.
- [29] Atsushi Ichino. Trilinear forms and the central values of triple product L -functions. *Duke Math. J.*, 145(2):281–307, 2008.
- [30] Henryk Iwaniec and Emmanuel Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.
- [31] H. Jacquet and D. Zagier. Eisenstein series and the Selberg trace formula. II. *Trans. Amer. Math. Soc.*, 300(1):1–48, 1987.
- [32] Dmitri Jakobson. Equidistribution of cusp forms on $\mathrm{PSL}_2(\mathbf{Z}) \backslash \mathrm{PSL}_2(\mathbf{R})$. *Ann. Inst. Fourier (Grenoble)*, 47(3):967–984, 1997.
- [33] Svetlana Katok and Peter Sarnak. Heegner points, cycles and Maass forms. *Israel J. Math.*, 84(1-2):193–227, 1993.
- [34] W. Kohnen and D. Zagier. Values of L -series of modular forms at the center of the critical strip. *Invent. Math.*, 64(2):175–198, 1981.
- [35] Pär Kurlberg, Lior Rosenzweig, and Zeév Rudnick. Matrix elements for the quantum cat map: fluctuations in short windows. *Nonlinearity*, 20(10):2289–2304, 2007.
- [36] Pär Kurlberg and Zeév Rudnick. On the distribution of matrix elements for the quantum cat map. *Ann. of Math. (2)*, 161(1):489–507, 2005.
- [37] Sheng-Chi Liu. A note on the variance of the distribution of Heegner points. *Int. J. Number Theory*, 8(2):525–529, 2012.
- [38] The LMFDB Collaboration. The L -functions and modular forms database. <http://www.lmfdb.org>, 2015. [Online; accessed 14 December 2015].
- [39] Wenzhi Luo. A note on the distribution of integer points on spheres. *Math. Z.*, 267(3-4):965–970, 2011.
- [40] Wenzhi Luo, Zeév Rudnick, and Peter Sarnak. The variance of arithmetic measures associated to closed geodesics on the modular surface. *J. Mod. Dyn.*, 3(2):271–309, 2009.
- [41] Wenzhi Luo and Peter Sarnak. Quantum ergodicity of eigenfunctions on $\mathrm{PSL}_2(\mathbf{Z}) \backslash \mathbf{H}^2$. *Inst. Hautes Études Sci. Publ. Math.*, (81):207–237, 1995.
- [42] Wenzhi Luo and Peter Sarnak. Mass equidistribution for Hecke eigenforms. *Comm. Pure Appl. Math.*, 56(7):874–891, 2003. Dedicated to the memory of Jürgen K. Moser.
- [43] Wenzhi Luo and Peter Sarnak. Quantum variance for Hecke eigenforms. *Ann. Sci. École Norm. Sup. (4)*, 37(5):769–799, 2004.
- [44] Greg Martin. Dimensions of the spaces of cusp forms and newforms on $\Gamma_0(N)$ and $\Gamma_1(N)$. *J. Number Theory*, 112(2):298–331, 2005.
- [45] Philippe Michel and Akshay Venkatesh. The subconvexity problem for GL_2 . *Publ. Math. Inst. Hautes Études Sci.*, (111):171–271, 2010.
- [46] Paul D. Nelson. Equidistribution of cusp forms in the level aspect. *Duke Math. J.*, 160(3):467–501, 2011.
- [47] Paul D. Nelson. Evaluating modular forms on Shimura curves. *Math. Comp.*, 84(295):2471–2503, 2015.
- [48] Paul D. Nelson. Analytic isolation of newforms of given level. preprint, 2016.
- [49] Paul D. Nelson. Microlocal lifts and quantum unique ergodicity on $\mathrm{GL}_2(\mathbb{Q}_p)$. preprint, 2016.
- [50] Paul D. Nelson. The spectral decomposition of $|\theta|^2$. preprint, 2016.
- [51] Paul D. Nelson, Ameya Pitale, and Abhishek Saha. Bounds for Rankin–Selberg integrals and quantum unique ergodicity for powerful levels. *J. Amer. Math. Soc.*, 27(1):147–191, 2014.
- [52] Stéphane Nonnenmacher. Anatomy of quantum chaotic eigenstates. In *Chaos*, volume 66 of *Prog. Math. Phys.*, pages 193–238. Birkhäuser/Springer, Basel, 2013.
- [53] Arnold Pizer. An algorithm for computing modular forms on $\Gamma_0(N)$. *J. Algebra*, 64(2):340–390, 1980.
- [54] Alexandru A. Popa. Central values of Rankin L -series over real quadratic fields. *Compos. Math.*, 142(4):811–866, 2006.
- [55] Kartik Prasanna. Integrality of a ratio of Petersson norms and level-lowering congruences. *Ann. of Math. (2)*, 163(3):901–967, 2006.

- [56] Kartik Prasanna. Arithmetic properties of the Shimura-Shintani-Waldspurger correspondence. *Invent. Math.*, 176(3):521–600, 2009. With an appendix by Brian Conrad.
- [57] Kartik Prasanna. On the Fourier coefficients of modular forms of half-integral weight. *Forum Math.*, 22(1):153–177, 2010.
- [58] Yannan Qiu. Periods of Saito-Kurokawa representations. *Int. Math. Res. Not. IMRN*, (24):6698–6755, 2014.
- [59] S. Rallis. On the Howe duality conjecture. *Compositio Math.*, 51(3):333–399, 1984.
- [60] Andre Reznikov. Rankin-Selberg without unfolding and bounds for spherical Fourier coefficients of Maass forms. *J. Amer. Math. Soc.*, 21(2):439–477, 2008.
- [61] P. Sarnak, P. Zhao, and A. b. M. Woodbury. The Quantum Variance of the Modular Surface. *ArXiv e-prints*, March 2013.
- [62] Roman Schubert. Upper bounds on the rate of quantum ergodicity. *Ann. Henri Poincaré*, 7(6):1085–1098, 2006.
- [63] Jean-Pierre Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.
- [64] Hideo Shimizu. Theta series and automorphic forms on GL_2 . *J. Math. Soc. Japan*, 24:638–683, 1972.
- [65] W. A. Stein et al. *Sage Mathematics Software (Version 6.7)*. The Sage Development Team, 2015. <http://www.sagemath.org>.
- [66] Marie-France Vignéras. *Arithmétique des algèbres de quaternions*, volume 800 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.
- [67] Thomas C. Watson. Rankin triple products and quantum chaos. [arXiv.org:0810.0425](https://arxiv.org/abs/0810.0425), 2008.
- [68] André Weil. Sur certains groupes d’opérateurs unitaires. *Acta Math.*, 111:143–211, 1964.
- [69] Don Zagier. Eisenstein series and the Selberg trace formula. I. In *Automorphic forms, representation theory and arithmetic (Bombay, 1979)*, volume 10 of *Tata Inst. Fund. Res. Studies in Math.*, pages 303–355. Tata Inst. Fundamental Res., Bombay, 1981.
- [70] S. Zelditch. Recent developments in mathematical Quantum Chaos. *ArXiv e-prints*, November 2009.
- [71] Steven Zelditch. Uniform distribution of eigenfunctions on compact hyperbolic surfaces. *Duke Math. J.*, 55(4):919–941, 1987.
- [72] Steven Zelditch. On the rate of quantum ergodicity. I. Upper bounds. *Comm. Math. Phys.*, 160(1):81–92, 1994.
- [73] Peng Zhao. Quantum variance of Maass-Hecke cusp forms. *Comm. Math. Phys.*, 297(2):475–514, 2010.

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